Mathematical theory for the incompressible Euler and Navier–Stokes equations in $\mathbb{R}^n$

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Contents

CHAPTER 1

Strong L^p solutions of the incompressible Navier–Stokes equations in $\mathbb{R}^n$

1.1 Introduction

The aim of this chapter is to provide detailed notes on Tosio Kato's article on the incompressible Navier–Stokes equation in the whole space [8]. We shall prove some results on the existence of local and global solutions of the Cauchy problem including some decay properties for the Navier–Stokes equation in the whole space domain $\mathbb{R}^n$, $n = 2, 3, 4, \ldots$:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f(x), \quad t > 0,\, x \in \mathbb{R}^n, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where $u_0 \in PL^n(\mathbb{R}^n) := PL^n(\mathbb{R}^n; \mathbb{R}^n)$ (sometimes denoted by just PL^n for short). Here we shall be concerned with solutions in the mild sense satisfying a corresponding integral equation and we only consider the inhomogeneous case where $f \equiv 0$. Now, the equations are scale invariant under the scaling transformation

$$(u_\lambda(x,t), p_\lambda(x,t)) \longrightarrow (\lambda u(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t)) \quad \text{for each } \lambda > 0.$$

Hence, this motivates our pursuit of establishing well-posedness for the Navier–Stokes equations in functions spaces whose norms preserve this scale invariance, since it should naturally lead to global-in-time well-posedness results. In doing so for the Navier–Stokes equations, however, some obstacles arise such as the continuity of the bilinear terms and the estimates for the heat semi-group, for instance. These issues are at the core of the celebrated problem on the global regularity and well-posedness of the Navier–Stokes equations.

1.2 Main results

Theorem 1.1. *Let $u_0 \in PL^n$. Then there exists a $T > 0$ and a unique solution u to (1.1) such that*

$$t^{(1-n/q)/2} u \in BC([0,T); PL^q) \text{ for } n \leqslant q \leqslant \infty, \tag{1.2}$$

$$t^{1-n/2q} \nabla u \in BC([0,T); PL^q) \text{ for } n \leqslant q < \infty, \tag{1.3}$$

both with value zero at $t = 0$ except when $n = q$ in (1.2), in which case $u(x,0) = u_0(x)$. Moreover, $u \in L^r((0,T_1); PL^q)$, $\frac{1}{r} = \frac{1}{2}\left(1 - \frac{n}{q}\right)$, $n < q < \frac{n^2}{n-2}$ for some $0 < T_1 \leqslant T$.

Theorem 1.2. *There is a $\lambda > 0$ such that if $\|u_0\|_n \leqslant \lambda$, then the solution u in Theorem 1.1 is global, i.e. we may take $T = T_1 = \infty$. In particular, $\|u(t)\|_q$ decays like $t^{-(1-n/q)/2}$ as $t \longrightarrow \infty$, including $q = \infty$, and $\|\nabla u(t)\|_q$ decays like $t^{-(1-n/2q)}$ as $t \longrightarrow \infty$, including $q = n$.*

1.3 Proof of Theorem 1.1

In this section, be shall prove Theorem 1.1 by first providing a complete outline of the proof by identifying the main key steps. We then fill in the details for each main step in the proof in the next section.

I. Reformulate (1.1) as an integral equation of the form

$$\boxed{u = e^{t\Delta} u_0 + Bu.}$$

Essentially, we obtain the desired well-posedness result of Theorem 1.1 by proving the existence and uniqueness of fixed point solutions to this integral equation.

(1) Let $e^{t\Delta}$ be the heat semi-group (convolution) operator on $\mathbb{R}^n$. Since $u_0 \in PL^n$, then $e^{t\Delta} u_0$ belongs in $BC([0,T); PL^n)$.

(2) Let $\mathbb{P}$ be the orthogonal projection from $L^2(\mathbb{R}^n)$ onto H, where

$$H := PL^2(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) \mid \nabla \cdot v = 0\}.$$

More precisely, the following Hodge decomposition holds: each vector field u in $L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ has a unique orthogonal decomposition:

$$u = w + \nabla p,$$

with the following properties:

(i) $\nabla \cdot w = 0$,

(ii) w and ∇p belong in $L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$,

(iii) $(w, \nabla p)_{L^2} = 0$,

(iv) for any multi-index β of the derivatives D^β, $|\beta| \geqslant 0$,

$$\|D^\beta u\|_2^2 = \|D^\beta w\|_2^2 + \|\nabla D^\beta p\|_2^2.$$

Thus, $\mathbb{P} : L^2(\mathbb{R}^n) \longrightarrow PL^2(\mathbb{R}^n)$ is defined by the projection map

$$\mathbb{P}u = \mathbb{P}(w + \nabla p) = w.$$

Observe that the divergence-free condition on w implies that we can recover p from u by the elliptic equation $\Delta p = \nabla \cdot u$. Once p is found, we obtain w from the relationship $w = u - \nabla p$.

One can show the projection operator is bounded in the dense subspace $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. By a standard density and continuity argument, we can extend this as a bounded operator $\mathbb{P} : L^p(\mathbb{R}^n) \mapsto PL^p(\mathbb{R}^n) := \{v \in L^p(\mathbb{R}^n) \mid \nabla \cdot v = 0\}$, $1 < p < \infty$. Alternatively, we can define the projection operator using Riesz transforms,

$$R_j = D_j(-\Delta)^{-1/2}, \; j = 1, 2, \ldots, n,$$

where $D_j = -i\dfrac{\partial}{\partial x_j}$. For an arbitrary vector field $v(x) = (v_1(x), v_2(x), \ldots, v_n(x))$, we set $z(x) = \displaystyle\sum_{k=1}^n (R_k v_k)(x)$ and defined the operator $\mathbb{P}$ by

$$(\mathbb{P}v)_j(x) = v_j(x) - (R_j z)(x) = \sum_{k=1}^n (\delta_{jk} - R_j R_k) v_k, \; j = 1, 2, \ldots, n.$$

Sometimes, this is written more concisely as

$$\mathbb{P} = Id - \nabla\Delta^{-1}\mathrm{div} = \mathbb{P} = Id + \nabla(-\Delta)^{-1}\mathrm{div}.$$

(3) Here we give a precise definition of the bilinear operator B.

(a) $F(u, v) := \mathbb{P}((u \cdot \nabla)v) = \mathbb{P}\nabla \cdot (u \otimes v)$ and $F(u) := F(u, u)$ for $u, v \in PL^q$.

(b) $B(u, v) := -\displaystyle\int_0^t e^{(t-s)\Delta} F(u(s), v(s))\, ds$ and $Bu(t) := B(u, u)$.

(c) Observe that we must define $F(u, v)$ in the distribution sense, i.e., for any smooth test function φ with compact support,

$$\langle \mathbb{P}(u \cdot \nabla v), \varphi \rangle := \mathbb{P}(u_i v_{i,j})(\varphi_j) = -\int_{\mathbb{R}^n} u_i v_j (\mathbb{P}\varphi_j)_{j,i}.$$

(4) Deriving the integral equation from (1.1). Note that since it is assumed the velocity field u is divergence-free, we have that $u = \mathbb{P}u$. Thus applying $\mathbb{P} : L^q(\mathbb{R}^n) \mapsto PL^q(\mathbb{R}^n)$ yields

$$\partial_t u - \Delta u = -F(u).$$

We can reformulate this non-linear, non-homogeneous heat equation into the desired integral equation via Duhamel's principle.

II. Establish the required estimates and find a suitable function space to set up the fixed point problem for the Integral equation.

(1) Basic Estimates for the Heat Semi-group Operator:

(a) $\|e^{t\Delta}u\|_q \leqslant C_n t^{-(n/p-n/q)/2}\|u\|_p$ with $1 < p \leqslant q \leqslant \infty$,

(b) $\|\nabla e^{t\Delta}u\|_q \leqslant C_n t^{-(1+n/p-n/q)/2}\|u\|_p$ with $1 < p \leqslant q < \infty$.

(c) $\|F(u,v)\|_p \leqslant C_n \|u\|_r \|\nabla v\|_s$ where $\dfrac{1}{p} = \dfrac{1}{r} + \dfrac{1}{s}$

Estimate (c) is just the Hölder inequality.

(2) Bilinear Estimates, i.e., estimates on the bilinear operator B.

Let $\frac{1}{p} = \frac{\alpha}{n} + \frac{\beta}{n}$. Applying the heat kernel estimates to B yield

$$\|Bu\|_q \leqslant C_n \int_0^t (t-s)^{-(\alpha+\beta-n/q)/2}\|u(s)\|_{n/\alpha}\|\nabla u(s)\|_{n/\beta}\, ds,$$

$$\|\nabla Bu\|_q \leqslant C_n \int_0^t (t-s)^{-(1+\alpha+\beta-n/q)/2}\|u(s)\|_{n/\alpha}\|\nabla u(s)\|_{n/\beta}\, ds.$$

Remark 1.1. *Here we assume $n/q \leqslant \alpha + \beta < n$ so that $q \geqslant p > 1$. Moreover, we must assume $\alpha + \beta - n/q < 2$ and $1 + \alpha + \beta - n/q < 2$ in order for the improper integrals in the Bilinear Estimates to converge.*

(3) Identify the appropriate space.

(a) Let

$$X_{q_1,q_2,T} := \left\{ u \mid t^{(1-n/q_1)/2}u \in BC([0,T]; PL^{q_1}),\ t^{1-n/2q_2}\nabla u \in BC([0,T]; PL^{q_2}) \right\}.$$

(b) Equip the function space $X_{q_1,q_2,T}$ with the norm

$$\|u\|_{q_1,q_2,T} := K(u,q_1,T) + K'(u,q_2,T),$$

where

$$K(u,q_1,T) := \sup_{0\leqslant t\leqslant T} t^{(1-n/q_1)/2}\|u(t)\|_{q_1}$$

and

$$K'(u,q_2,T) := \sup_{0\leqslant t\leqslant T} t^{1-n/2q_2}\|\nabla u(t)\|_{q_2}.$$

(4) Estimates on the bilinear operator B.

Given $q \geqslant n$ and for any fixed $0 < \delta < 1$,

$$\boxed{\|Bu - Bv\|_{q,n,T} \leqslant C\{\|u\|_{n/\delta,n,T} + \|v\|_{n/\delta,n,T}\}\|u - v\|_{n/\delta,n,T}} \tag{1.4}$$

where $C = C(n,q,\delta)$ depends on n, q, and δ, $\dfrac{n}{q} \leqslant \delta + 1 \leqslant n$. We note that we establish this contraction estimate using the bilinear estimates on B with $\beta = 1$, and $\alpha = \delta$.

III. Use the Contraction Mapping Principle to obtain the local existence and uniqueness of mild solutions to the integral equation

$$u = \Phi u := e^{t\Delta}u_0 + Bu \quad \text{satisfying conditions (1.2)–(1.3).}$$

In proving Kato's results, we shall only consider two cases: when $n < q$ or when $n = q$ in (1.2) where we always fix $q = n$ in (1.3).

(1) **Case:** $\frac{n}{q} < 1$.

We set up a fixed point problem.

(a) $\Phi : \overline{B}(0, 1/4C) \subset X_{q,n,T} \longrightarrow \overline{B}(0, 1/4C)$ for $T > 0$ sufficiently small.

(b) $\|\Phi u - \Phi v\|_{q,n,T} \leqslant \frac{1}{2}\|u - v\|_{q,n,T}$ for all $u,\ v \in \overline{B}(0, 1/4C)$.

(2) **Case:** $q = n$.

Similarly as the previous case, however, to show $u \in BC([0,T); PL^n)$, we set up a fixed point problem with slight modifications.

(a) Define $\Omega_1 := \overline{B}(0,R) \subset BC([0,T]; PL^n)$ and $\Omega_2 := \overline{B}(0, 1/8C) \subset X_{\frac{n}{\delta},n,T}$, and consider the intersection subspace $\Omega := \Omega_1 \cap \Omega_2$ equipped with the norm

$$\|\cdot\| := \|\cdot\|_{n,n,T} + \|\cdot\|_{\frac{n}{\delta},n,T}$$

for a fixed $0 < \delta < 1$ and the constant $C = \max\{C_1, C_2\}$ where C_i come from the contraction estimates below.

(b) Show Φ maps Ω_1 to itself and Ω_2 to itself.

(c) Contraction estimates: For any $u, v \in \Omega$, we have the following:

- $\|Bu - Bv\|_{n,n,T} \leqslant C_1 \left(\|u\|_{\frac{n}{\delta},n,T} + \|v\|_{\frac{n}{\delta},n,T}\right) \|u - v\|_{\frac{n}{\delta},n,T} \leqslant \frac{1}{4}\|u - v\|_{\frac{n}{\delta},n,T}$.
- $\|Bu - Bv\|_{\frac{n}{\delta},n,T} \leqslant C_2 \left(\|u\|_{\frac{n}{\delta},n,T} + \|v\|_{\frac{n}{\delta},n,T}\right) \|u - v\|_{\frac{n}{\delta},n,T} \leqslant \frac{1}{4}\|u - v\|_{\frac{n}{\delta},n,T}$.

Thus

$$\boxed{\|\Phi u - \Phi v\| \leqslant \frac{1}{2}\|u - v\|}$$

for all $u, v \in \Omega$.

(d) By the Contraction mapping principle, there exists a unique solution $u \in \Omega$ to the integral equation $u = \Phi u$.

Remark 1.2. *The above estimates on B are a crucial ingredients in the proofs of Theorems 1.1 and 1.2 and illustrates the need to consider intersection of subspaces and connects the two cases when $q > n$ and $q = n$. Namely, for the case $q = n$, we are not able to close the estimates with respect to the function space $X_{n,n,T}$. Fortunately, we can close the estimates in the subspace $X_{\frac{n}{\delta},n,T}$ thereby circumventing the difficulty by taking the intersection of the two spaces and applying the fixed point argument to the integral equation valued in this new intersection space.*

Remark 1.3. *Although we only consider two cases for the value of q, it is simple to prove the actual theorem for all given values of q; that is, we can prove the integrability for the unique strong solution for all indices of $q \geqslant n$. More specifically, we can adopt the same fixed point argument as above but in the space*

$$\mathcal{X}_T \doteq \bigcap_{q_1,q_2 \geqslant n} X_{q_1,q_2,T}$$

equipped with the norm

$$\| \cdot \|_{\mathcal{X}_T} \doteq \sum_{q_1,q_2 \geqslant n} \| \cdot \|_{q_1,q_2,T}.$$

Here, we allow $q_1 = \infty$.

1.4 Detailed calculations in the proof of Theorem 1.1

Let us first prove the heat kernel estimates II.(2). Recall that

$$e^{t\Delta} u = G_t * u = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u(y,t)\, dy.$$

Using Young's inequality, we have

$$\begin{aligned}
\|e^{t\Delta} u\|_q = \|G_t * u\|_q &\leqslant \|G_t\|_r \|u\|_p \quad \text{where } 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p} \\
&\leqslant \frac{1}{(4\pi t)^{n/2}} \left(\int_{\mathbb{R}^n} e^{-\frac{r|x|^2}{4t}}\, dx \right)^{1/r} \|u\|_p \\
&\underbrace{=}_{w=x/\sqrt{4t}} \frac{1}{(4\pi t)^{n/2}} \left(\int_{\mathbb{R}^n} e^{-r|w|^2} (4t)^{n/2}\, dw \right)^{1/r} \|u\|_p \\
&\leqslant C_n t^{-(n-n/r)/2} \|u\|_p \\
&\leqslant C_n t^{-(n/p-n/q)/2} \|u\|_p.
\end{aligned}$$

Similarly, the estimate on $\nabla e^{t\Delta}u$ follows similarly by first absorbing the derivative onto to the heat kernel, i.e.,

$$\|\nabla e^{t\Delta}u\|_q = \|\nabla G_t * u\|_q \leqslant \|\nabla G_t\|_r \|u\|_p \leqslant C_n t^{-(1+n-n/r)/2}\|u\|_p \leqslant C_n t^{-(1+n/p-n/q)/2}\|u\|_p.$$

Now let us prove the Bilinear estimates II.(2) using the heat kernel estimates. Recall that

$$\frac{1}{p} = \frac{\alpha}{n} + \frac{\beta}{n}.$$

Then

$$\begin{aligned}
\|Bu\|_q := \|Bu\|_q &\leqslant \left\| \int_0^t e^{(t-s)\Delta} F(u(s))\, ds \right\|_q \\
&\leqslant \int_0^t \|e^{(t-s)\Delta} F(u(s))\|_q \, ds \\
&\leqslant \int_0^t (t-s)^{-(n/p-n/q)/2} \|F(u(s))\|_p \, ds \\
&\leqslant \int_0^t (t-s)^{-(\alpha+\beta-n/q)/2} \|u(s)\|_{n/\alpha} \|\nabla u(s)\|_{n/\beta}\, ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\nabla Bu\|_q &\leqslant \left\| \int_0^t \nabla e^{(t-s)\Delta} F(u(s))\, ds \right\|_q \\
&\leqslant \int_0^t \|\nabla e^{(t-s)\Delta} F(u(s))\|_q \, ds \\
&\leqslant \int_0^t (t-s)^{-(1+n/p-n/q)/2} \|F(u(s))\|_p \, ds \\
&\leqslant \int_0^t (t-s)^{-(1+\alpha+\beta-n/q)/2} \|u(s)\|_{n/\alpha} \|\nabla u(s)\|_{n/\beta}\, ds.
\end{aligned}$$

Let us now prove (1.4). Recall that $\beta = 1$ and $\alpha = \delta \in (0,1)$. Using the bilinear estimates and a change of variables, we compute

$$\begin{aligned}
\|Bu(t)-Bv(t)\|_q &\leqslant \int_0^t (t-s)^{-(1+\delta-n/q)/2}\|(u-v)\cdot\nabla u + v\cdot\nabla(u-v)\|_p\,ds \\
&\leqslant C\int_0^t (t-s)^{-(1+\delta-n/q)/2}\Big\{\|u-v\|_{n/\delta}\|\nabla u\|_n + \|v\|_{n/\delta}\|\nabla(u-v)\|_n\Big\}\,ds \\
&\leqslant C\int_0^t (t-s)^{-(1+\delta-n/q)/2} s^{-(1-\delta/2)} s^{-(1-1/2)}\times \\
&\quad\times\Big\{K(u-v,n/\delta,T)K'(u,n,T)+K(v,n/\delta,T)K'(u-v,n,T)\Big\}\,ds \\
&\underbrace{\leqslant}_{\tilde{s}=s/t} C\int_0^1 (t(1-\tilde{s}))^{-(1+\delta-n/q)/2}(\tilde{s}t)^{-(2-\delta)/2}t\,d\tilde{s}\times \\
&\quad\times\Big\{K(u-v,n/\delta,T)K'(u,n,T)+K(v,n/\delta,T)K'(u-v,n,T)\Big\} \\
&\leqslant Ct^{-(1+\delta-n/q+2-\delta-2)/2}\int_0^1 (1-s)^{-(1+\delta-n/q)/2}s^{-(2-\delta)/2}\,ds\times \\
&\quad\times\Big\{K(u-v,n/\delta,T)K'(u,n,T)+K(v,n/\delta,T)K'(u-v,n,T)\Big\} \\
&\leqslant Ct^{-(1-n/q)/2}\Big\{\int_0^1 (1-s)^{-(1+\delta-n/q)/2}s^{-(2-\delta)/2}\,ds\Big\}\times \\
&\quad\times\Big\{K(u-v,n/\delta,T)K'(u,n,T)+K(v,n/\delta,T)K'(u-v,n,T)\Big\} \qquad (1.5)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\nabla Bu(t)-\nabla Bv(t)\|_n &\leqslant \int_0^t (t-s)^{-(2+\delta-n/n)}\Big\{\|u-v\|_{n/\delta}\|\nabla u\|_n + \|v\|_{n/\delta}\|\nabla(u-v)\|_n\Big\}\,ds \\
&\leqslant C\int_0^t (t-s)^{-(1+\delta)/2}s^{-(1-\delta)/2}s^{-(1-1/2)}\,ds \\
&\quad\times\Big\{K(u-v,n/\delta,T)K'(u,n,T)+K(v,n/\delta,T)K'(u-v,n,T)\Big\} \\
&\leqslant Ct^{-1/2}\Big\{\int_0^1 (1-s)^{-(1+\delta)/2}s^{-(2-\delta)/2}\,ds\Big\}\times \\
&\quad\times\Big\{K(u-v,n/\delta,T)K'(u,n,T)+K(v,n/\delta,T)K'(u-v,n,T). \qquad (1.6)
\end{aligned}$$

Combining the two estimate (1.5) and (1.6) and taking the supremum over t yields the desired contraction estimates.

Let us now show the existence and uniqueness of a local-in-time fixed point solution to $u = \Phi u$. Here we shall consider $n \leqslant q < \infty$ in (1.2) and $q = n$ in (1.3). The other cases for n and q are treated similarly.

Remark 1.4. *As in the previous remark, the difficulty that stems from the contraction estimate is that we must impose the condition* $\alpha = \delta \in (0,1)$ *for the case* $q = n$*, i.e., when* $1 = n/q$*. Namely, the estimates do not hold when* $\delta = 1$ *for this case since the improper integrals in the bilinear estimates diverge (recall we had to assume that* $1 + \alpha + \beta - n/q = 1 + \delta + 1 - 1 < 2$*).*

The next result concerns the estimates on the heat kernel acting on the initial data $u_0 \in PL^n(\mathbb{R}^n)$*. These estimates are required for the fixed point argument, however, we require a splitting of the initial data into a small part and a smooth compactly supported part. As* $L^n(\mathbb{R}^n)$ *is a critical space, however, the scaling invariance is negated by the splitting of the initial condition and so rescaling cannot be used to obtain global well-posedness. We resolve this issue and deduce a global existence result but at the expense of restricting ourselves to sufficiently small initial data.*

We digress somewhat to clarify the notion of a critical function space for the incompressible Navier–Stokes equations in $\mathbb{R}^3$.

Definition 1.1. *A translation or shift invariant Banach space of tempered distributions* X *is called a* **critical space** *for the Navier–Stokes equations if its norm is invariant under the action of the scaling* $f(x) \longrightarrow \lambda f(\lambda x)$ *for any* $\lambda > 0$*. In other words, we require that*

$$X \hookrightarrow \mathcal{S}'$$

and that for any $f \in X$

$$\|f(\cdot)\|_X = \|\lambda f(\lambda \cdot - x_0)\|_X \text{ for all } \lambda > 0, \text{ and for all } x_0 \in \mathbb{R}^n.$$

Remark 1.5. *Some examples of critical spaces for the three-dimensional Navier–Stokes equations are the following:*

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_q^{-1+3/q,\infty}(\mathbb{R}^3) \hookrightarrow BMO^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}_\infty^{-1,\infty}(\mathbb{R}^3),$$

where $q \geqslant 3$ *and the homogeneous Besov space* $\dot{B}_\infty^{-1,\infty}(\mathbb{R}^3)$ *is the largest critical space. Here, we shall focus on the Lebesgue space,* $L^3(\mathbb{R}^3)$ *in great detail, especially in next chapter. Generally speaking, the Lebesgue space* $L^n(\mathbb{R}^n)$*,* $p = n$ *is a critical space for the Navier–Stokes equations in* $\mathbb{R}^n$*.*

More generally, it turns out that for q and $r \in [1, \infty]$, the homogeneous Besov spaces $\dot{B}_q^{-1+n/q,r}(\mathbb{R}^n)$ are critical spaces for the Navier–Stokes equations in $\mathbb{R}^n$, and this follows from the next result.

Proposition 1.1. *Let* $s \in \mathbb{R}$ *and* $q, r \in [1, \infty]$*. Then there exists a constant* $C > 0$*, depending only on* s*, such that*

$$C^{-1}\lambda^{s-n/q}\|u\|_{\dot{B}_q^{-1+n/q,r}(\mathbb{R}^n)} \leqslant \|u(\lambda \cdot - x_0)\|_{\dot{B}_q^{-1+n/q,r}(\mathbb{R}^n)} \leqslant C\lambda^{s-n/q}\|u\|_{\dot{B}_q^{-1+n/q,r}(\mathbb{R}^n)}$$

for all $u \in \dot{B}_q^{-1+n/q,r}(\mathbb{R}^n)$*.*

Surprisingly, showing that $\dot{B}_\infty^{-1,\infty}(\mathbb{R}^3)$ is the largest critical space is very simple to prove.

Proposition 1.2. *The homogeneous Besov space $\dot{B}_\infty^{-1,\infty}(\mathbb{R}^n)$ is the largest critical space for the Navier–Stokes equations.*

Proof. Let $X \hookrightarrow S'(\mathbb{R}^n)$ be some critical space, i.e., assume that for any $(\lambda, x_0) \in (0,\infty) \times \mathbb{R}^n$,

$$\|u(\lambda \cdot -x_0)\|_X = \lambda^{-1}\|u\|_X.$$

Now, we want to show $X \hookrightarrow \dot{B}_\infty^{-1,\infty}(\mathbb{R}^n)$. Since X is continuously embedded in $S'(\mathbb{R}^n)$, we have that

$$|\langle u, e^{-|\cdot|^2}\rangle| \leqslant C\|u\|_X.$$

Then for any $x \in \mathbb{R}^n$ with $x_0 = -x$ and using the substitution $y \longrightarrow \lambda y + x$, we obtain

$$\begin{aligned}\frac{1}{\lambda^n}\int_{\mathbb{R}^n} u(y)e^{-|x-y|^2/\lambda^2}\,dy &= \int_{\mathbb{R}^n} u(\lambda y - x_0)e^{-|y|^2}\,dy = \langle u(\lambda \cdot -x_0), e^{-|\cdot|^2}\rangle \\ &\leqslant C\|u(\lambda \cdot -x_0)\|_X = C\lambda^{-1}\|u\|_X.\end{aligned}$$

From this, we use dilation with $\lambda = \sqrt{t}$ to get

$$\sqrt{t}\|e^{t\Delta}u\|_{L^\infty(\mathbb{R}^n)} \leqslant C\|u\|_X \ \text{ for all } \ t > 0,$$

which implies for $q = \infty$,

$$\|u\|_{\dot{B}_q^{-1+n/q,\infty}(\mathbb{R}^n)} = \|u\|_{\dot{B}_\infty^{-1,\infty}(\mathbb{R}^n)} \doteq \sup_{t>0}\sqrt{t}\|e^{t\Delta}u\|_{L^\infty(\mathbb{R}^3)} \leqslant C\|u\|_X$$

This completes the proof. ∎

Remark 1.6. *In the later chapters, we give a concise overview of the homogeneous Besov spaces with the help of the Littlewood–Paley decomposition.*

We return to the details of the proof of Theorem 1.1. The following lemma examines the required step of splitting the initial data. This is precisely the mechanism responsible for requiring our smallness restriction on u_0.

Lemma 1.1 (Splitting of initial data)**.** *If $q > n$, then*

- $\|t^{(1-n/q)/2}e^{t\Delta}a\|_q \longrightarrow 0$ *as* $t \longrightarrow 0$,
- $\|t^{1/2}\nabla e^{t\Delta}a\|_q \longrightarrow 0$ *as* $t \longrightarrow 0$.

Proof. To prove these two properties, we split the initial condition using a density argument. Indeed, for each $\varepsilon > 0$, we decompose u_0 into $u_0 = a_1 + a_2$, where a_1 is an L^n function with small L^n-norm, i.e., $\|a_1\|_n \leqslant \varepsilon$ and a_2 is a smooth and compactly supported function. Let $\delta = \frac{n}{q} \in (0,1)$ and fix some $l \in (\delta, 1)$, then the heat kernel estimates imply

$$\begin{aligned}
\|t^{(1-\delta)/2} e^{t\Delta} u_0\|_q &= \|t^{(1-\delta)/2} e^{t\Delta} u_0\|_{n/\delta} \\
&\leqslant \|t^{(1-\delta)/2} e^{t\Delta} a_1\|_{n/\delta} + \|t^{(1-\delta)/2} e^{t\Delta} a_2\|_{n/\delta} \\
&\leqslant C_n t^{(1-\delta)/2} t^{-(n/n - n/(n/\delta))/2} \|a_1\|_n + t^{(1-l+l-\delta)/2} \|e^{t\Delta} a_2\|_{n/\delta} \\
&\leqslant C_n \varepsilon + t^{(1-l)/2} t^{(l-\delta)/2} t^{-(n/(n/l) - n/(n/\delta))/2} \|a_2\|_{n/l} \\
&\leqslant C_n \Big\{ \varepsilon + t^{(1-l)/2} \|a_2\|_{n/l} \Big\}.
\end{aligned}$$

The second property is verified in a similar fashion. $\square$

The previous estimate implies that if our initial datum u_0 is sufficiently small in $L^n(\mathbb{R}^n)$, then $e^{t\Delta} u_0$ will remain small in global time. Now we ready to exploit these smallness properties in setting up our fixed point argument. In other words, we can now choose $T > 0$ sufficiently small so that $\|e^{t\Delta} u_0\|_{n/\delta,n,T} \leqslant 1/8C$. For $u \in \overline{B}(0, 1/4C) \subset X_{n/\delta,n,T}$,

$$\begin{aligned}
\|\Phi u\|_{n/\delta,n,T} &\leqslant \|e^{t\Delta} u_0\|_{n/\delta,n,T} + \|Bu\|_{n/\delta,n,T} \\
&\leqslant \frac{1}{8C} + \frac{1}{2}\frac{1}{4C} = \frac{1}{4C} \text{ from section III.2.c.}
\end{aligned}$$

So Φ maps $\overline{B}(0, 1/4C)$ into itself. Using the contraction estimates,

$$\|Bu - Bv\|_{n/\delta,n,T} \leqslant \frac{1}{2} \|u - v\|_{n/\delta,n,T},$$

so by the Contraction mapping principle, there exists a unique $u \in \overline{B}(0, 1/4C)$ such that $u = \Phi u$.

Let $\frac{n}{q} = 1$. We will show $u \in BC([0,T]; PL^n)$ with the aide of the previous case. First we define the following subspaces:

- $\Omega_1 := \overline{B}(0, R) \subset BC([0,T]; PL^n)$ where R is to be later specified,
- $\Omega_2 := \overline{B}(0, 1/4C) \subset X_{n/\delta,n,T}$,
- Consider the closed subspace $\Omega := \Omega_1 \cap \Omega_2$ equipped with the norm

  $$\| \cdot \|_\Omega := \| \cdot \|_{n,n,T} + \| \cdot \|_{n/\delta,n,T}.$$

  Hence, Ω is complete under the induced norm topology.

From the contraction estimates III.2.c, for any $u, v \in \Omega$,

- $\|Bu(t) - Bv(t)\|_{n/\delta,n,T} \leqslant C_1 \left\{ \|u\|_{n/\delta,n,T} + \|v\|_{n/\delta,n,T} \right\} \|u(t) - v(t)\|_{n/\delta,n,T}$,

- $\|Bu(t) - Bv(t)\|_{n,n,T} \leqslant C_2 \left\{\|u\|_{n/\delta,n,T} + \|v\|_{n/\delta,n,T}\right\} \|u(t) - v(t)\|_{n/\delta,n,T}$.

Now set $R = \sup_{0 \leqslant t \leqslant T} \|e^{t\Delta} u_0\|_n + 1/C$ with $C = \max\{C_1, C_2\}$, then

$$\|\Phi u\|_{\Omega_1} \leqslant \|e^{t\Delta} u_0\|_{\Omega_1} + \|Bu\|_{\Omega_1} \leqslant \sup_{0 \leqslant t \leqslant T} \|e^{t\Delta} u_0\|_m + 1/16C < R,$$

and

$$\|\Phi u\|_{\Omega_2} \leqslant \|e^{t\Delta} u_0\|_{\Omega_2} + \|Bu\|_{\Omega_2} \leqslant \frac{1}{8C} + C\left(\frac{1}{16C^2} + \frac{1}{16C^2}\right) \leqslant 1/4C.$$

Thus Φ maps Ω into itself and the contraction estimates imply it is a contraction mapping on Ω. Hence, the Contraction mapping principle implies the existence and uniqueness of an element $u \in \Omega$ such that $u = \Phi u$.

1.5 Proof of Theorem 1.2

Global well-posedness: Observe that the bilinear estimates hold for any time $T > 0$ and that we can find a suitably small $\lambda > 0$ such that if $\|u_0\|_n \leqslant \lambda$, then the bilinear estimates and the norms of $e^{t\Delta} u_0$ are independent of T. For instance, let us verify this for the latter statement. Recall from the heat kernel estimates, we get

$$\|e^{t\Delta} u_0\|_{n/\delta} \leqslant C t^{-(1-\delta)/2} \|u_0\|_n,$$

and

$$\|\nabla e^{t\Delta} u_0\|_n \leqslant C t^{-1/2} \|u_0\|_n.$$

Hence, we may take $T = \infty$ in the space $X_{n/\delta,n,T}$ and

$$\|e^{t\Delta} u_0\|_{n/\delta,n,\infty} \leqslant C \|u_0\|_n$$

and our fixed point argument applies accordingly. That is, we may find an absolute constant λ such that if $\|u_0\|_n \leqslant \lambda$ then the contraction mapping principle implies global-in-time existence and uniqueness of a solution in the usual subspace of $C([0,\infty); L^n(\mathbb{R}^n))$. The remaining asymptotic results follow from Theorem 1.1.

Remark 1.7. *The global existence and uniqueness result of Kato presented here holds for a smaller subspace of $BC([0,\infty); L^n(\mathbb{R}^n))$, yet it was not known at that time whether this solution is unique in $BC([0,\infty; L^n(\mathbb{R}^n))$. Indeed, it was proved to be unique in this class later in [5]. In addition, basic regularity theory implies this unique solution is also smooth for $t > 0$ (cf. [9] for proofs of these regularity and uniqueness results).*

CHAPTER 2

Extension of Kato's theorem for the Navier–Stokes equations in critical Lebesgue and Besov spaces

In this chapter, we extend Kato's existence theorems of Chapter 1 by following the work of Cannone [3]. Namely, we obtain a global well-posedness result for the incompressible Navier–Stokes equations in the critical space $L^n(\mathbb{R}^n)$ with solenoidal initial data $u_0 \in L^n(\mathbb{R}^n)$ which are small in the homogeneous Besov space $\dot{B}_q^{1-n/q,\infty}(\mathbb{R}^n)$. In particular, the types of initial data that exhibit these properties include those which are sufficiently oscillating.

For the sake of simplicity, we only consider the three-dimensional incompressible Navier–Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, & t > 0,\ x \in \mathbb{R}^3, \\ \nabla \cdot u = 0, & \\ u(x,0) = u_0(x), & \end{cases} \tag{2.1}$$

We reduce this problem into the mild integral equation:

$$u(t) = e^{t\Delta}u_0 + B(u,u), \tag{2.2}$$

where the bilinear operator is defined as

$$B(u,v) = -\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes v)\, ds. \tag{2.3}$$

We mention that the equivalence between (2.1) and (2.2) holds under classical solutions and even weaker notions of suitable solutions (cf. [9] for further details on the matter). Let us also mention the familiar idea of showing the existence of fixed point solutions to this integral equation in the setting of Lebesgue spaces. As stated later in Section 2.1.1, the key idea is to identify the functional space in which to set up the fixed point argument, however, the bi-continuity of the bilinear operator required in applying Picard's theorem breaks down in the critical space $C([0,T); PL^3(\mathbb{R}^3))$. In fact, in his unpublished doctorial thesis, F. Oru proved the non-continuity of the bilinear operator not only in this case but also in $C([0,T); L^{(3,q)}(\mathbb{R}^3))$ where $q \in [1,\infty)$ and $L^{(3,q)}(\mathbb{R}^3)$ is a Lorentz space. It is interesting that the limiting case where $q = \infty$ is completely different, since Y. Meyer in [10] showed that the bi-continuity holds in $C([0,T); L^{(3,\infty)}(\mathbb{R}^3))$. Nevertheless, the issue of continuity in the class $C([0,T); PL^3(\mathbb{R}^3))$ was circumvented in Chapter 1 by carefully choosing an appropriate auxiliary subspace of the critical space in which the bi-continuity property holds. The results in this chapter essentially adopts the same ideas but weakens Kato's original assumptions on the initial data. On the other hand, the bi-continuity of the bilinear operator holds easily for the super-critical spaces, $C([0,T); L^q(\mathbb{R}^3))$ where $q > 3$. However, as a caveat, this continuity is only good enough to establish local well-posedness and it remains to be known if global well-posedness holds. For completeness sake, we state and prove the existence results for both critical and super-critical spaces below.

To elucidate the differences between Cannone's theorem with that of Kato's—at the expense of repeating ourselves—we state both theorems. We can state a simplified version of Kato's Theorem (Theorem 1.1 in Chapter 1) as follows.

Note that from this point on, we set $\alpha = \alpha(q) \doteq 1 - 3/q$.

Theorem 2.1 (Kato)**.** *Let $q \in (3,6]$ be fixed. Then there exists an absolute constant $\delta > 0$, such that if $u_0 \in L^3(\mathbb{R}^3)$, $\|u_0\|_3 < \delta$, and $\nabla \cdot u_0 = 0$ (in the distribution sense), then there exists a global mild solution of the Navier–Stokes equations in $C([0,\infty); L^3(\mathbb{R}^3))$. Moreover, this solution is the only one such that*

$$u(t,x) \in C([0,\infty); PL^3(\mathbb{R}^3)),$$

$$t^{\alpha/2} u(t,x) \in C([0,\infty); PL^q(\mathbb{R}^3)),$$

and

$$\lim_{t \longrightarrow 0} t^{\alpha/2} \|u(t)\|_q = 0.$$

Theorem 2.2 (Cannone)**.** *Let $q \in (3,6]$ be fixed. Then there exists an absolute constant $\delta > 0$ such that if $u_0 \in L^3(\mathbb{R}^3)$, $\|u_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$, and $\nabla \cdot u_0 = 0$ (in the distribution sense), then there exists a global mild solution of the Navier–Stokes equations in $C([0,\infty); PL^3(\mathbb{R}^3))$. Moreover, this solution is the only one such that*

$$u(t,x) \in C([0,\infty); PL^3(\mathbb{R}^3)),$$

$$t^{\alpha/2} u(t,x) \in C([0,\infty); PL^q(\mathbb{R}^3)),$$

and

$$\lim_{t \longrightarrow 0} t^{\alpha/2} \|u(t)\|_q = 0.$$

2.1 Some preliminaries and the main result

In this section, we provide the necessary background for obtaining the main existence theorem, then we prove the main results of this chapter.

2.1.1 Bicontinuous bilinear operators and Picard's Theorem

Proposition 2.1 (Picard Contraction Principle)**.** *Let X be an abstract Banach space with norm $\|\cdot\|_X$ and $B : X \times X \longrightarrow X$ a bilinear operator. Suppose that $B : X \times X \longrightarrow X$ is bicontinuous, i.e., for any $x_1, x_2 \in X$,*

$$\|B(x_1, x_2)\|_X \leqslant \eta \|x_1\|_X \|x_2\|_X,$$

then for any $y \in X$ such that $4\eta\|y\|_X < 1$, the equation

$$x = y + B(x, x)$$

has a solution $x \in X$. In particular, the solution satisfies $\|x\|_X \leqslant 2\|y\|_X$ and is the only one for which $\|x\| < \frac{1}{2\eta}$.

Proof. The proof is quite standard, but we provide it for the reader's convenience. Set $R := 2\|y\|_X$ and define the map $\Phi(x) : X \longrightarrow X$ such that

$$\Phi(x) = y + B(x, x).$$

Then

$$\begin{aligned}\|\Phi(x)\|_X &\leqslant \|y\|_X + \|B(x,x)\|_{X\times X} \leqslant \|y\|_X + \eta\|x\|_X\|x\|_X \\ &\leqslant \|y\|_X + \eta(2\|y\|_X)^2 \leqslant \|y\|_X + 4\eta\|y\|_X^2 \\ &\leqslant \|y\|_X(1 + \underbrace{4\eta\|y\|_X}_{<1}) \leqslant 2\|y\|_X \leqslant R.\end{aligned}$$

Thus, this implies that $\Phi(x)$ maps $\overline{B}_R(0) \subset X$, the closed ball of radius R centered at the origin, to itself. Moreover, for any x_1 and x_2 in $\overline{B}_R(0)$,

$$\begin{aligned}\|\Phi(x_1) - \Phi(x_2)\|_X &\leqslant \|B(x_1, x_1) - B(x_2, x_2)\|_X \\ &\leqslant \|B(x_1 - x_2, x_1) + B(x_2, x_1 - x_2)\|_X \\ &\leqslant \|B(x_1 - x_2, x_1)\|_X + \|B(x_2, x_1 - x_2)\|_X \\ &\leqslant \eta\|x_1\|_X\|x_1 - x_2\|_X + \eta\|x_2\|_X\|x_1 - x_2\|_X \\ &\leqslant \underbrace{2R\,\eta}_{=\ 4\eta\|y\|_X} \|x_1 - x_2\|_X\end{aligned}$$

Thus, $\Phi : \overline{B}_R(0) \longrightarrow \overline{B}_R(0)$ is a strict contraction and, by Picard iteration, implies there exists a unique fixed point $x \in \overline{B}_R(0)$ such that $\Phi(x) = x$, i.e., $x = y + B(x, x)$.

Now, suppose that $\tilde{x} \in X$ is another fixed point solution for which $\|x\|_X, \|\tilde{x}\|_X < 1/2\eta$. Then we have

$$\begin{aligned}\|x - \tilde{x}\|_X &= \|\Phi(x) - \Phi(\tilde{x})\|_X \leqslant \|B(x, x) - B(\tilde{x}, \tilde{x})\|_X \\ &\leqslant \|B(x - \tilde{x}, x) + B(\tilde{x}, x - \tilde{x})\|_X \\ &\leqslant \eta\|x\|_X\|x - \tilde{x}\|_X + \eta\|\tilde{x}\|_X\|x - \tilde{x}\|_X \\ &\leqslant \eta(\|x\|_X + \|\tilde{x}\|_X)\|x - \tilde{x}\|_X \\ &< \eta(1/2\eta + 1/2\eta)\|x - \tilde{x}\|_X \\ &\leqslant \|x - \tilde{x}\|_X,\end{aligned}$$

which implies $x = \tilde{x}$. This completes the proof. $\square$

2.1.2 The Littlewood–Paley decomposition

Let us describe the Littlewood–Paley decomposition in $\mathbb{R}^3$. We arbitrarily choose a function φ in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ and whose Fourier transform $\widehat{\varphi}$ satisfies

$$0 \leqslant \widehat{\varphi} \leqslant 1, \ \widehat{\varphi}(\xi) = 1 \text{ if } |\xi| \leqslant 3/4, \ \widehat{\varphi}(\xi) = 0 \text{ if } |\xi| \geqslant 3/2,$$

and let

$$\begin{aligned}\psi(x) &= 8\varphi(2x) - \varphi(x), \\ \varphi_j(x) &= 2^{3j}\varphi(2^j x), \ j \in \mathbb{Z}, \\ \psi_j(x) &= 2^{3j}\psi(2^j x), \ j \in \mathbb{Z}.\end{aligned}$$

Denote by S_j and Δ_j, respectively, the convolution operators with φ_j and ψ_j. The set $\{S_j, \Delta_j\}_{j\in\mathbb{Z}}$ is the Littlewood–Paley decomposition for which

$$I = S_0 + \sum_{j\geqslant 0} \Delta_j. \tag{2.4}$$

Note that this decomposition does not depend on the choice of φ. Moreover, for any given tempered distribution $f \in \mathcal{S}'(\mathbb{R}^3)$,

$$f = \lim_{j\longrightarrow\infty} S_0 f + \sum_{j\geqslant 0} \Delta_j f. \tag{2.5}$$

In particular, the identity,

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f,$$

is to be understood modulo polynomials, i.e., $f \in \mathcal{S}'(\mathbb{R}^3)\backslash\mathcal{P}$.

Let us describe this decomposition in a more precise manner. We start with the following theorem.

Theorem 2.3. *For all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^3)$, we have*

$$f = S_N f + \sum_{j \geqslant N} \Delta_j f \in \mathcal{S}'(\mathbb{R}^3).$$

This equality is called the Littlewood–Paley decomposition of the distribution f. If, moreover, $\lim_{N\to-\infty} S_N f = 0$ in $\mathcal{S}'(\mathbb{R}^3)$, then the equality

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f$$

is called the homogeneous Littlewood–Paley decomposition of f.

Definition 2.1. *We define the space of tempered distributions vanishing at infinity as the space $\mathcal{S}'_0(\mathbb{R}^3)$ of distributions so that $\lim_{N\to-\infty} S_N f = 0$ in $\mathcal{S}'(\mathbb{R}^3)$.*

For more general tempered distributions, we cannot recover them from their homogeneous Littlewood–Paley decomposition but modulo polynomials:

Lemma 2.1. *For all $f \in \mathcal{S}'(\mathbb{R}^3)$, there is an integer N and a sequence of polynomials $\{P_j\}_{j\in\mathbb{Z}}$ of degree $\leqslant N$ so that*

$$\sum_{j \in \mathbb{Z}} \Delta_j f + P_j$$

converges to f in $\mathcal{S}'(\mathbb{R}^3)$. Hence, the equality,

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f,$$

holds in $\mathcal{S}'(\mathbb{R}^3)/\mathbb{C}[X_1, X_2, \ldots, X_{d=3}]$.

Remark 2.1. *We can see that $\mathcal{S}'_0(\mathbb{R}^3) = \mathcal{S}'(\mathbb{R}^3)\backslash\mathcal{P}$. The interest in decomposing a tempered distribution into a sum of dyadic blocks $\Delta_j f$, whose support in Fourier space is localized in a corona, comes from the favorable behavior of these blocks with respect to differential operations. This can be illustrated by the celebrated Bernstein's Lemma in $\mathbb{R}^3$.*

Lemma 2.2 (Bernstein)**.** *Let $1 \leqslant p \leqslant q \leqslant \infty$ and $k \in \mathbb{N}$, then*

$$\sup_{|\alpha|=k} \|D^\alpha f\|_p \simeq R^k \|f\|_p,$$

and

$$\|f\|_q \lesssim R^{3(\frac{1}{p}-\frac{1}{q})} \|f\|_p,$$

whenever $f \in \mathcal{S}'(\mathbb{R}^3)$ whose Fourier transform $\hat{f}(\xi)$ is supported in the corona $|\xi| \simeq R$.

In case the function has support in a ball (e.g. $S_j f$), then the following version holds.

Lemma 2.3. *Let $1 \leqslant p \leqslant q \leqslant \infty$ and $k \in \mathbb{N}$, then*

$$\sup_{|\alpha|=k} \|D^\alpha f\|_p \simeq R^k \|f\|_p,$$

and

$$\|f\|_q \lesssim R^{3(\frac{1}{p}-\frac{1}{q})} \|f\|_p,$$

whenever $f \in \mathcal{S}'(\mathbb{R}^3)$ whose Fourier transform $\hat{f}(\xi)$ is supported in the ball $|\xi| \lesssim R$.

2.1.3 The Besov spaces $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$

Here, we shall provide a brief characterization of homogeneous Besov spaces via Littlewood-Paley theory. There is a natural motivation for examining well-posedness for the Navier–Stokes equations in the homogeneous Besov spaces, since these function spaces are appropriate for scale invariant equations.

Definition 2.2. *Let q be fixed in $1 \leqslant q \leqslant \infty$ and $\alpha \in \mathbb{R}$. A tempered distribution $f \in \mathcal{S}_0'(\mathbb{R}^3)$ belongs in the Besov space $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ if*

$$\|f\|_{\dot{B}_q^{-\alpha,\infty}} = \sup_{j \in \mathbb{Z}} 2^{-j\alpha} \|\Delta_j f\|_q$$

is finite. The following lemma provides an equivalent characterization of the Besov space $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ in terms of the heat semi-group. This is useful since our estimates below involves the heat semi-group.

Lemma 2.4. *Let q be fixed in $1 \leqslant q \leqslant \infty$ and $\alpha > 0$. For any tempered distribution $f \in \mathcal{S}_0'(\mathbb{R}^3)$, the following norms,*

$$(a)\ \sup_{j \in \mathbb{Z}} 2^{-j\alpha} \|\Delta_j f\|_q,$$

$$(b)\ \sup_{j \in \mathbb{Z}} 2^{-j\alpha} \|S_j f\|_q,$$

$$(c)\ \sup_{t \geqslant 0} t^{\alpha/2} \|e^{t\Delta} f\|_q,$$

$$(d)\ \sup_{t \geqslant 0} \|e^{t\Delta} f\|_{\dot{B}_q^{-\alpha,\infty}},$$

are equivalent.

The next lemma is on the embedding properties between $L^3(\mathbb{R}^3)$ and these Besov spaces.

Lemma 2.5. *Let q_1 and q_2 be two fixed constants in $3 \leqslant q_1 \leqslant q_2 \leqslant \infty$, and set $\alpha_1 = 1 - 3/q_1$ and $\alpha_2 = 1 - 3/q_2$. Then*

$$L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{q_1}^{-\alpha_1,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}_{q_2}^{-\alpha_2,\infty}(\mathbb{R}^3).$$

Proof. From the Bernstein's inequalities, we deduce that

$$2^{-j\alpha_2} \|\Delta_j f\|_{q_2} \lesssim 2^{-j\alpha_1} \|\Delta_j f\|_{q_1} \lesssim \|\Delta_j f\|_3 \lesssim \|f\|_3.$$

The desired result follows immediately. $\square$

We point out that the following chain of continuous embeddings are strict. For instance, the function $|x|^{-1}$ belongs in $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$, however, $|x|^{-1} \notin L^3(\mathbb{R}^3)$.

2.1.4 The proof of Theorem 2.2

Let $X = G$ be the Banach space of functions $v(t,x)$ satisfying

$$v(t,x) \in C([0,\infty); PL^3(\mathbb{R}^3)), \tag{2.6}$$

$$t^{\alpha/2} v(t,x) \in C([0,\infty); PL^q(\mathbb{R}^3)), \tag{2.7}$$

and

$$\lim_{t \longrightarrow 0} t^{\alpha/2} \|v(t)\|_q = 0. \tag{2.8}$$

equipped with the norm

$$\|v\|_G := \sup_{t>0} \|v(t)\|_{\dot{B}_q^{-\alpha,\infty}} + \sup_{t>0} t^{\alpha/2} \|v(t)\|_q. \tag{2.9}$$

Now, let us state and prove some important lemmas that will play significant roles in the proof of the main theorem.

Lemma 2.6. *If $u_0 \in PL^3(\mathbb{R}^3)$, then $e^{t\Delta}u_0 \in G$.*

Proof. We will prove that $e^{t\Delta}u_0$ satisfies (2.6)–(2.8). First, it is clear that the heat semi-group operator $u_0 \longrightarrow e^{t\Delta}u_0$ preserves the divergence-free condition. Secondly, recall from Lemma 2.4 that the norm of $e^{t\Delta}u_0$ in G is equivalent to $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ norm of u_0 and that $L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$, i.e.,

$$\|u_0\|_{\dot{B}_q^{-\alpha,\infty}} \lesssim \|e^{t\Delta}u_0\|_G \lesssim \|u_0\|_{\dot{B}_q^{-\alpha,\infty}} \lesssim \|u_0\|_3.$$

Hence, $e^{t\Delta}u_0 \in C([0,\infty); PL^3(\mathbb{R}^3))$. Moreover, the heat kernel estimates of Chapter 1 imply

$$\|t^{\alpha/2} e^{t\Delta}u_0\|_q \leqslant t^{\alpha/2} t^{-(3/3-3/q)/2} \|u_0\|_3 = t^{(\alpha-\alpha)/2} \|u_0\|_3 = \|u_0\|_3.$$

Thus,

$$\sup_{t>0} \|t^{\alpha/2} e^{t\Delta}u_0\|_q \leqslant \|u_0\|_3,$$

which implies $t^{\alpha/2} e^{t\Delta}u_0 \in C([0,\infty); PL^3(\mathbb{R}^3))$. Similarly, from the heat kernel estimates and since the heat semi-group $e^{t\Delta}$ is strongly continuous in $L^p(\mathbb{R}^n)$ where $1 < p < \infty$, we have

$$\|t^{\alpha/2} e^{t\Delta}u_0\|_q \leqslant t^{\alpha/2} \|e^{t\Delta}u_0\|_q \longrightarrow 0 \text{ as } t \longrightarrow 0.$$

□

Remark 2.2. *The equivalence in norm, or more specifically, the estimate*

$$\|e^{t\Delta}u_0\|_G \lesssim \|u_0\|_{\dot{B}_q^{-\alpha,\infty}},$$

plays a very important role in obtaining the global well-posedness of solutions. More precisely, this implies that if we choose a sufficiently small initial data in $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$, then $\|e^{t\Delta}u_0\|_G$ also remains sufficiently small and Picard's theorem yields global solutions. In Kato's proof presented in Chapter 1, we applied a splitting procedure on $u_0 \in L^3(\mathbb{R}^3)$ to apply Picard's theorem. Consequently, we obtained global mild solutions provided we had sufficiently small initial data in $L^3(\mathbb{R}^3)$, instead.

Lemma 2.7. *The bilinear operator $B(u, v)(t)$ defined by*

$$B(u, v)(t) := -\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes v)(s)\, ds, \tag{2.10}$$

is bicontinuous in $G \times G \longrightarrow G$.

Proof. For the sake of simplicity, we shall prove the bicontinuity of the bilinear operator $B(u, v)(t)$ in the scalar case, which can be expressed as

$$B(f, g)(t) = -\int_0^t \frac{1}{(t-s)^{-2}}\Theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * (fg)(s)\, ds,$$

where $f = f(t, x)$ and $g = g(t, x)$ are scalar fields in G and $\Theta = \Theta(x)$ has Fourier transform given by

$$\hat{\Theta}(\xi) = |\xi|e^{-|\xi|^2},$$

and as such, is an analytic function which is $\mathcal{O}(|x|^{-4})$ at infinity. In other words, we are treating $e^{t\Delta}\mathbb{P}\nabla \cdot (u \otimes v)$ as a single convolution operator unlike what was done in Chapter 1. By Young's inequality (here, the condition $q \leqslant 6$ appears) followed by the substitution $x/\sqrt{t-s} \longrightarrow x$ in order to compute $\|\Theta\|_r$, we obtain

$$\begin{aligned}\|B(f, g)(t)\|_3 &\leqslant \int_0^t \frac{1}{(t-s)^2}\left\|\Theta\left(\frac{\cdot}{\sqrt{t-s}}\right)\right\|_{r_1}\|f(s)g(s)\|_{q/2}\, ds \\ &\leqslant \left(\int_0^t (t-s)^{-2+3/(2r_1)}\|f(s)g(s)\|_{q/2}\, ds\right)\|\Theta\|_{r_1}\end{aligned}$$

where $1 + \frac{1}{3} = \frac{1}{r_1} + \frac{1}{q/2}$. Then, Hölder's inequality implies

$$\begin{aligned}\|B(f, g)(t)\|_3 &\leqslant \left(\int_0^t (t-s)^{-2+3/(2r_1)}s^{-\alpha}\, ds\right)\|\Theta\|_r \sup_{t>0} t^{\alpha/2}\|f(t)\|_q \sup_{t>0} t^{\alpha/2}\|g(t)\|_q \\ &\lesssim \sup_{t>0} t^{\alpha/2}\|f(t)\|_q \cdot \sup_{t>0} t^{\alpha/2}\|g(t)\|_q \end{aligned} \tag{2.11}$$

Similarly, Young's inequality implies

$$\|B(f, g)(t)\|_q \leqslant \left(\int_0^t (t-s)^{-2+3/(2r_2)}\|f(s)g(s)\|_{q/2}\, ds\right)\|\Theta\|_{r_2},$$

where $1 + \frac{1}{q} + \frac{1}{r_2} + \frac{2}{q}$. By Hölder's inequality, we obtain

$$
\begin{aligned}
\|B(f,g)(t)\|_q &\leqslant \left(\int_0^t (t-s)^{-2+3/2r_2} s^{-\alpha}\, ds\right) \|\Theta\|_{r_2} \sup_{t>0} t^{\alpha/2}\|f(t)\|_q \cdot \sup_{t>0} t^{\alpha/2}\|g(t)\|_q \\
&\underbrace{\leqslant}_{\tilde{s}=s/t} \left(\int_0^t t^{-2+3/(2r_2)}(1-\tilde{s})^{-2+3/(2r_2)} t^{-\alpha} s^{-\alpha} t\, d\tilde{s}\right) \|\Theta\|_{r_2} \sup_{t>0} t^{\alpha/2}\|f(t)\|_q \cdot \sup_{t>0} t^{\alpha/2}\|g(t)\|_q \\
&\lesssim t^{-1+3/(2r_2)-\alpha} \sup_{t>0} t^{\alpha/2}\|f(t)\|_q \cdot \sup_{t>0} t^{\alpha/2}\|g(t)\|_q \\
&\lesssim t^{-\alpha/2} \sup_{t>0} t^{\alpha/2}\|f(t)\|_q \cdot \sup_{t>0} t^{\alpha/2}\|g(t)\|_q,
\end{aligned}
$$

where we used the fact that $-1 + 3/(2r_2) = \alpha/2$. This implies

$$
t^{\alpha/2}\|B(f,g)(t)\|_q \lesssim \sup_{t>0} t^{\alpha/2}\|f(t)\|_q \cdot \sup_{t>0} t^{\alpha/2}\|g(t)\|_q. \tag{2.12}
$$

Hence, the estimates (2.11) and (2.12), in general, imply that

$$
\|B(u,v)(t)\|_G \lesssim \|u\|_G\|v\|_G \quad \text{for any} \quad u, v \in G.
$$

Now, the the bilinear estimates can be easily applied to show that

$$
\lim_{t\longrightarrow 0} t^{\alpha/2}\|B(f,g)(t)\|_q = 0,
$$

whenever

$$
\lim_{t\longrightarrow 0} t^{\alpha/2}\|f(t)\|_q = \lim_{t\longrightarrow 0} t^{\alpha/2}\|g(t)\|_q = 0.
$$

Furthermore, we can also show that if the latter conditions hold, then

$$
\lim_{t\longrightarrow 0} \|B(f,g)(t)\|_3 = 0.
$$

In particular, this convergence property is an important ingredient in our proof of the main theorem since it guarantees any solution $u(t,x) \in G$ of the integral equation (2.2) with solenoidal initial data $u_0 \in L^3(\mathbb{R}^3)$ is unique in G and tends to u_0 in the strong topology of $L^3(\mathbb{R}^3)$. □

Proof of Theorem 2.2. The theorem follows directly from Proposition 2.1 and the previous lemmas. □

2.1.5 Proof of the local existence theorem in super-critical space

For completeness sake, we establish the bilinear estimates for the critical space $C([0,\infty); L^q(\mathbb{R}^3))$, $q > 3$. Hence, the existence and uniqueness of mild solutions follow from Proposition 2.1 without resorting to any auxiliary subspace as in the critical case. We show the bicontinuity of the bilinear operator $B : X \times X \longrightarrow X$ where $X = C([0,\infty); L^q(\mathbb{R}^3))$. As before, we establish the bilinear estimates by considering the scalar case of the bilinear operator. By Young's inequality followed by the substitution $x/\sqrt{t-s} \longrightarrow x$, we obtain

$$\begin{aligned}
\|B(f,g)(t)\|_q &\leqslant \int_0^t \frac{1}{(t-s)^2} \left\| \Theta\left(\frac{\cdot}{\sqrt{t-s}}\right) \right\|_r \|f(s)g(s)\|_{q/2}\, ds \\
&\leqslant \left(\int_0^t (t-s)^{-2+3/(2r)} \|f(s)g(s)\|_{q/2}\, ds \right) \|\Theta\|_r
\end{aligned}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{q/2}$, i.e., $\frac{1}{q} + \frac{1}{r} = 1$. Then, Hölder's inequality implies

$$\begin{aligned}
\|B(f,g)(t)\|_q &\leqslant \left(\int_0^t (t-s)^{-2+3/(2r)}\, ds \right) \|\Theta\|_r \sup_{t>0} \|f(t)\|_q \cdot \sup_{t>0} \|g(t)\|_q \\
&\leqslant \left(\int_0^t (t-s)^{-2+3/2-3/(2q)}\, ds \right) \|\Theta\|_r \sup_{t>0} \|f(t)\|_q \cdot \sup_{t>0} \|g(t)\|_q \\
&\lesssim \left(\int_0^t (t-s)^{-1/2-3/(2q)}\, ds \right) \sup_{t>0} \|f(t)\|_q \cdot \sup_{t>0} \|g(t)\|_q \\
&\lesssim \sup_{t>0} \|f(t)\|_q \cdot \sup_{t>0} \|g(t)\|_q, \qquad (2.13)
\end{aligned}$$

where the integral in the estimate converges since $\frac{1}{2} + \frac{3}{2q} < 1$ whenever $q > 3$. Hence, we have shown the following.

Lemma 2.8. *Let $3 < q \leqslant \infty$ be fixed. For any $T > 0$ and any functions $f(t), g(t) \in C([0,T); L^q(\mathbb{R}^3))$, then the bilinear term $B(f,g)(t)$ also belongs to $C([0,T); L^3(\mathbb{R}^3))$ and we have*

$$\sup_{0<t<T} \|B(f,g)(t)\|_q \lesssim \frac{T^{(1-3/q)/2}}{1-3/q} \sup_{0<t<T} \|f(t)\|_q \sup_{0<t<T} \|g(t)\|_q.$$

As a consequence, we obtain the following existence result.

Theorem 2.4. *Let $3 < q \leqslant \infty$ be fixed. For any $u_0 \in PL^q(\mathbb{R}^3)$, there exists a $T = T(\|u_0\|_q)$ such that the Navier–Stokes equations has a unique solution in $C([0,T); PL^q(\mathbb{R}^3))$.*

Remark 2.3. *It is still an open question whether the solution in the super-critical setting are global and the non-invariance of the L^q norm for $q \neq 3$ ensures that such a global result would not depend on the size of the initial data, $\|u_0\|_q$.*

Remark 2.4. *Notice that the local well-posedness for the Navier–Stokes equations still holds for the super-critical case when $q = \infty$; however, some modifications are needed since $L^\infty(\mathbb{R}^3)$ is not separable and therefore the heat semi-group is not strongly continuous as $t \longrightarrow 0$. (cf. Section 3.3 in [4] for further details).*

2.2 The Cannone–Meyer–Planchon Theorem

This section states and proves the Cannone–Meyer–Planchon Theorem, which is a global well-posedness result for the Navier–Stokes equations and is closely related to the previous well-posedness results of T. Kato and M. Cannone. This theorem, however, achieves global well-posedness for small initial data in the typical scale invariant space $\dot{B}_q^{-1+3/q,\infty}(\mathbb{R}^3)$ but, instead, the fixed point argument is formulated in an auxiliary subspace of $L^\infty([0,\infty); \dot{B}_q^{-1+3/q,\infty}(\mathbb{R}^3))$. Thus, it suffices to verify the continuity of the bilinear terms under this setting, and this argument relies on ideas from Littlewood–Paley theory and the smoothing effect of the heat kernel.

First we recall an important property from [1], which describes the action of the semigroup of the heat flow on distributions with Fourier transforms supported on an annulus.

Lemma 2.9 (Lemma 2.4 in [1])**.** *Let $\mathcal{C}$ be an annulus. Then there exist positive constants c and C such that for any $q \in [1,\infty]$ and any couple (t,λ) of positive real numbers, we have that*

$$\text{if } \operatorname{supp} \widehat{u} \subset \lambda\mathcal{C} \text{ then, } \|e^{t\Delta}u\|_{L^q} \leqslant Ce^{-ct\lambda^2}\|u\|_{L^q}.$$

From this, we have that $\|\dot{\Delta}_j e^{t\Delta}u_0\|_{L^q} \leqslant Ce^{-ct2^{2j}}\|\dot{\Delta}_j u_0\|_{L^q}$. Integrating this in time yields

$$\|\dot{\Delta}_j e^{t\Delta}u_0\|_{L^1(L^q)} \leqslant \frac{C}{2^{2j}} 2^{-j(-1+3/q)}\|u_0\|_{\dot{B}_q^{-1+3/q,\infty}}. \tag{2.14}$$

This observation leads to the following definition.

Definition 2.3. *For $1 \leqslant q \leqslant \infty$, we denote by E_q the space of functions u in $L^\infty([0,\infty); \dot{B}_q^{-1+3/q,\infty}(\mathbb{R}^n))$ for which*

$$\|u\|_{E_q} \doteq \sup_j 2^{j(-1+3/q)}\|\dot{\Delta}_j u\|_{L^\infty(L^q)} + \sup_j 2^{2j}2^{j(-1+3/q)}\|\dot{\Delta}_j u\|_{L^1(L^p)} < \infty.$$

Note that estimate (2.14) implies that $\|e^t\Delta u_0\|_{E_q} \leqslant C\|u_0\|_{\dot{B}_q^{-1+3/q,\infty}}$.

We have the following main result.

Theorem 2.5. *Let $q \in [1,\infty)$. There exists a constant δ such that the integral equation* (2.2) *has a unique solution u in $B_{2\delta}(0) \subset E_q$ whenever $\|u_0\|_{\dot{B}_q^{-1+3/q,\infty}} \leqslant \delta$.*

This follows from the standard fixed point argument provided we have the following bilinear estimate on E_q.

Lemma 2.10. *There exists a constant C such that for any $q \in [1,\infty)$,*

$$\|B(u,v)\|_{E_q} \leqslant C \cdot q\|u\|_{E_q}\|v\|_{E_q}.$$

Here $B(u,v)$ denotes the usual bilinear operator from the Navier–Stokes equations.

Proof of Lemma 2.10. We omit the proof but refer the reader to Chapter 5 page 234 in [1]. $\square$

CHAPTER 3

On the breakdown of smooth solutions for the 3-D incompressible Euler equations

3.1 Introduction

This chapter provides notes from Beale, Kato, and Majda's [2] result on the finite-time blowup of smooth solutions to the incompressible Euler equations in $\mathbb{R}^3$ (an analogous result does hold for the incompressible Navier–Stokes equations). We shall consider the initial value problem to the three dimensional incompressible Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0, \quad t > 0,\, x \in \mathbb{R}^3, \\ \nabla \cdot u = 0. \end{cases} \tag{3.1}$$

It turns out that a necessary condition for the finite-time blowup of classical solutions of (3.1) is directly related to the time integral of the supremum norm of the vorticity, $\omega := \nabla \times u$. Namely, if a solution of the Euler equations is initially "smooth" and loses its regularity at some later time, then the supremum of the vorticity necessarily grows without bound as time approaches a critical value T_*. An equivalent reformulation of this statement is that if the vorticity remains bounded, then a smooth solution persists.

3.2 Main Results

Theorem 3.1. *Let u be a solution of the Euler equations* (3.1) *and*

$$u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1}). \tag{3.2}$$

Suppose that there is a time T_* such that the solution cannot be continued in the class (3.2) *to $T = T_*$ and assume T_* is the first such time. Then*

$$\int_0^{T_*} \|\omega(t)\|_{L^\infty}\, dt = \infty, \tag{3.3}$$

and in particular,

$$\limsup_{t \to T_*^-} \|\omega(t)\|_{L^\infty} = \infty.$$

An immediate consequence of this theorem is the following result.

Corollary 3.1. *For some solution of the Euler equations* (3.1)*, suppose there are constants M_0 and T_* so that on any interval $[0, T]$ of existence of the solution in the class* (3.2)*, with $T < T_*$, the vorticity satisfies the a priori estimate*

$$\int_0^{T} \|\omega(t)\|_{L^\infty}\, dt \leqslant M_0. \tag{3.4}$$

Then the solution can be continued in the class (3.2) *to the interval $[0, T_*]$.*

Proof of Theorem 3.1. First, we show that the assumptions imply that

$$\limsup_{t \to T_*} \|u(t)\|_{H^s} = \infty. \tag{3.5}$$

To see this, assume the contrary. That is, $\|u(t)\|_{H^s} \leqslant C_0$ for some positive constant C_0 and all $t < T_*$. By the local well-posedness, we can start a solution at any time t_1 with initial value $u(t_1)$, and this solution will be regular for $t_1 \leqslant t \leqslant t_1 + T_0(C_0)$, with T_0 independent of t_1. If $t_1 > T_* - T_0$, then we have extended the original solution past the time T_*, which contradicts our choice for T_*.

To prove the theorem, we claim that if

$$\int_0^{T_*} \|\omega(t)\|_{L^\infty}\, dt \equiv M_0 < \infty, \tag{3.6}$$

then

$$\|u(t)\|_{H^s} \leqslant C_0 \ \text{ for } \ t < T_*,$$

for some positive constant C_0, thus contradicting (3.5).

Step 1: We estimate $\omega(t)$ in L^2. Taking the curl in (3.1) leads to the vorticity equation

$$\omega_t + u \cdot \nabla u = \omega \cdot \nabla u. \tag{3.7}$$

Recall the important property that

$$((u \cdot \nabla) w, w) = 0 \ \text{ at least for } \ w \in H^1,$$

which follows from integration by parts and the divergence-free condition on the velocity u. Thus, if we multiply (3.7) by ω and integrate, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 = (\omega \cdot \nabla u, \omega). \tag{3.8}$$

The velocity u is determined from the vorticity ω by the relation

$$u = -\nabla \times (\nabla^{-1}\omega).$$

Therefore, the Fourier transforms of ∇u and ω satisfy $\widehat{\nabla u}(\xi) = S(\xi)\widehat{\omega}(\xi)$ where S is a matrix bounded independent of ξ. Thus, we have

$$\|\nabla u\|_{L^2} \leqslant C\|\omega\|_{L^2}.$$

By inserting this into (3.8) and using the Cauchy–Schwarz inequality, we arrive at

$$\frac{d}{dt}\|\omega(t)\|_{L^2}^2 \leqslant 2Cm(t)\|\omega(t)\|_{L^2}^2, \tag{3.9}$$

where $m(t) = \|\omega(t)\|_{L^\infty}$, so that

$$\|\omega(t)\|_{L^2} \leqslant \|\omega(0)\|_{L^2} \exp\left\{C\int_0^t m(\tau)\, d\tau\right\},$$

or

$$\|\omega(t)\|_{L^2} \leqslant M_1\|\omega(0)\|_{L^2} \tag{3.10}$$

with $M_1 = \exp(CM_0)$.

Step 2: Next, we derive energy estimates for (3.1) in terms of $\|\nabla u\|_{L^\infty}$.

Let α be a multi-index with $|\alpha| \leqslant s$. Set $v = D_x^\alpha u$. By applying the usual differential operator D_x^α to (3.1), we obtain for v the equation

$$v_t + u \cdot \nabla v + \nabla q = -F, \tag{3.11}$$

where $q = D^\alpha p$ and

$$F = D^\alpha(u \cdot \nabla u) - u \cdot \nabla D^\alpha u.$$

We estimate F by recalling the well-known elementary inequality

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2} \leqslant C\left(\|f\|_{H^s}\|g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|g\|_{H^{s-1}}\right)$$

by taking $f = u$ and $g = \nabla u$ so that

$$\|D^\alpha(u \cdot \nabla u) - u \cdot D^\alpha \nabla u\|_{L^2} \leqslant C\left(\|u\|_{H^s}\|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|\nabla u\|_{H^{s-1}}\right).$$

From this, we have that

$$\|F\|_{L^2} \leqslant C\|\nabla u\|_{L^\infty}\|u\|_{H^s}.$$

Applying standard methods in obtaining energy estimates for (3.11) yields

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^s}^2 \leqslant C\|\nabla u\|_{L^\infty}\|u\|_{H^s}^2,$$

which yields

$$\|u(t)\|_{H^s} \leqslant \|u(0)\|_{H^s} \exp\Big\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau\Big\}. \tag{3.12}$$

Step 3: We complete the proof by invoking the following time-independent estimate for $\|\nabla u\|_{L^\infty}$ in terms of bounds on ω and slight dependence on a higher norm of u,

$$\|\nabla u\|_{L^\infty} \leqslant C\Big\{1 + (1+\log^+\|u\|_{L^3})\|\omega\|_{L^\infty} + \|\omega\|_{L^2}\Big\}, \tag{3.13}$$

where $\log^+ a = \log a$ if $a \geqslant 1$ and $\log^+ a = 0$ otherwise. By virtue of (3.6) and (3.10), we can express inequality (3.13) as

$$\|\nabla u\|_{L^\infty} \leqslant C\Big\{1 + m(t)\log(\|u\|_{L^3} + e)\Big\}. \tag{3.14}$$

Here and below, C denotes a positive constant depending on M_0 and T_*.

Set $y(t) = \|u(t)\|_{H^s} + e$. By combining (3.12) and (3.14), we obtain

$$y(t) \leqslant y(0)\exp\Big\{C\int_0^t (1 + m(\tau)\log y(\tau))\,d\tau\Big\},$$

and if $z(t) = \log y(t)$,

$$z(t) \leqslant z(0) + C\int_0^t (1 + m(\tau)z(\tau))\,d\tau.$$

Hence, by Gronwall's inequality, $z(t)$ is bounded by a constant depending on M_0, T_*, and $\|u_0\|_{H^s}$. This completes the proof of the theorem. □

CHAPTER 4

Littlewood–Paley theory and the Besov and Triebel–Lizorkin spaces

In this Chapter, we introduce the Littlewood-Paley decomposition, the paradifferential calculus and related topics. We use these tools for defining and characterizing the Besov and Triebel–Lizorkin spaces. For more detailed accounts of the theory presented in this chapter, the reader is referred to [1, 6, 7, 9].

4.1 Littlewood–Paley theory

4.1.1 Bernstein-type Lemmas

Lemma 4.1. *Let $\mathcal{C}$ be an annulus and B a ball. A constant C such that for any non-negative integer k, any $p, q \in [1, \infty]$ with $q \geqslant p$, and any function $u \in L^p(\mathbb{R}^n)$, we have*

$$\text{If supp } \widehat{u} \subset \lambda B, \text{ then } \|D^k u\|_{L^q} \doteq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leqslant C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p},$$

$$\text{If supp } \widehat{u} \subset \lambda \mathcal{C}, \text{ then } C^{-(k+1)} \lambda^k \|u\|_{L^p} \leqslant \|D^k u\|_{L^p} \leqslant C^{k+1} \lambda^k \|u\|_{L^p}.$$

Lemma 4.2. *Let $\mathcal{C}$ be an annulus, $m \in \mathbb{R}$, and $k = 2[1 + n/2]$[1] Let σ be a k-times differentiable function on $\mathbb{R}^n \backslash \{0\}$ such that for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant k$, there exists a constant C_α such that*

$$\text{for all } \xi \in \mathbb{R}^n, \ |\partial^\alpha \sigma(\xi)| \leqslant C_\alpha |\xi|^{m-|\alpha|}.$$

Moreover, there exists a constant C, depending only on the constants C_α, such that for any $p \in [1, \infty]$ and any $\lambda > 0$, we have, for any function $u \in L^p$ with Fourier transform supported in $\lambda\mathcal{C}$,

$$\|\sigma(D)u\|_{L^p} \leqslant C\lambda^m \|u\|_{L^p},$$

[1] $[i]$ denotes the integer part of i.

where we define

$$\sigma(D)u \doteq \mathcal{F}^{-1}(\sigma\widehat{u}).$$

4.1.2 Dyadic partition of unity

Proposition 4.1. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^n \,|\, 3/4 \leqslant |\xi| \leqslant 8/3\}$. There exist radial functions $\widehat{\chi}$ and $\widehat{\varphi}$, valued in the interval $[0,1]$, belonging respectively to $\mathcal{D}(B_{4/3}(0))$ and $\mathcal{D}(\mathcal{C})$, and such that*

$$\textit{for all } \xi \in \mathbb{R}^n, \ \widehat{\chi}(\xi) + \sum_{j\geqslant 0} \widehat{\varphi}(2^{-j}\xi) = 1, \tag{4.1}$$

$$\textit{for all } \xi \in \mathbb{R}^n \backslash \{0\}, \ \sum_{j\geqslant 0} \widehat{\varphi}(2^{-j}\xi) = 1, \tag{4.2}$$

$$\textit{if } |j-j'| \geqslant 2, \ \textit{ then } \ \textit{supp } \widehat{\varphi}(2^{-j}\cdot) \cap \textit{supp } \widehat{\varphi}(2^{-j}\cdot) = \varnothing, \tag{4.3}$$

$$\textit{if } j \geqslant 1, \ \textit{ then } \ \textit{supp } \chi \cap \textit{supp } \widehat{\varphi}(2^{-j}\cdot) = \varnothing, \tag{4.4}$$

the set $\tilde{\mathcal{C}} \doteq B_{2/3}(0) + \mathcal{C}$ is an annulus, and we have

$$\textit{if } |j-j'| \geqslant 5, \ \textit{ then } \ 2^{j'}\tilde{\mathcal{C}} \cap 2^j\mathcal{C} = \varnothing. \tag{4.5}$$

Proof. Fix α to be in $(1, 4/3)$ and denote by $\mathcal{C}' = \{\xi \in \mathbb{R}^n \,|\, \alpha^{-1} \leqslant |\xi| \leqslant 2\alpha\} \subset \mathcal{C}$. Choose a smooth radial function θ valued in $[0,1]$, supported in $\mathcal{C}$, and $\theta \equiv 1$ in a neighborhood of $\mathcal{C}'$. The important point is the following: for any couple (j, j'), we have

$$|j-j'| \geqslant 2 \Longrightarrow 2^{j'}\mathcal{C} \cap 2^j\mathcal{C} = \varnothing. \tag{4.6}$$

Clearly, if $2^{j'}\mathcal{C} \cap 2^j\mathcal{C} \neq \varnothing$ and $j' \geqslant j$, then $2^{j'} \times 3/4 \leqslant 4 \times 2^{j+1}/3$, which implies that $j'-j \leqslant 1$. Now, let

$$S(\xi) = \sum_{j\in\mathbb{Z}} \theta(2^{-j}\xi).$$

Thanks to (4.6), this sum is locally finite on the set $\mathbb{R}^n\backslash\{0\}$. Thus, the function S is smooth on $\mathbb{R}^n\backslash\{0\}$. As $\alpha > 1$, we have

$$\bigcup_{j\in\mathbb{Z}} 2^j\mathcal{C}' = \mathbb{R}^n\backslash\{0\}.$$

Since the function θ is non-negative and has value 1 near $\mathcal{C}'$, it follows from the above covering property that the function S is positive. We claim that $\widehat{\varphi} \doteq \theta/S$ is suitable. Indeed, it is clear that $\widehat{\varphi}$ belongs to $\mathcal{D}(\mathcal{C})$ and that the function $1 - \sum_{j\geqslant 0}\widehat{\varphi}(2^{-j}\cdot)$ is smooth by (4.6). Moreover, as supp $\theta \subset \mathcal{C}$, we have that for $|\xi| \geqslant 4/3$, then

$$\sum_{j\geqslant 0} \widehat{\varphi}(2^{-j}\xi) = 1. \tag{4.7}$$

Thus, by setting

$$\widehat{\chi}(\xi) = 1 - \sum_{j \geqslant 0} \widehat{\varphi}(2^{-j}\xi),$$

we obtain the identities (4.1) and (4.3). Identity (4.4) is an immediate consequence of (4.6) and (4.7).

We now prove (4.5). By definition, the annulus $\tilde{\mathcal{C}} = \{\xi \in \mathbb{R}^n \,|\, 3/4 - 2/3 \leqslant |\xi| \leqslant 8/3 + 2/3\}$, i.e., it has center 0, small radius 1/12, and large radius 10/3. Then, it turns out that

$$2^k \tilde{\mathcal{C}} \cap 2^j \mathcal{C} \neq \varnothing \Rightarrow 3/4 \times 2^j \leqslant 2^k \times 10/3 \text{ or } 1/12 \times 2^k \leqslant 2^j \times 8/3$$

from which the identity (4.5) follows. This completes the proof. □

From this point on, we fix two function χ and φ whose Fourier transforms $\widehat{\chi}$ and $\widehat{\varphi}$ satisfy the properties of the previous proposition. We now define the homogeneous and non-homogeneous dyadic blocks and the low frequency cut-off operators.

Definition 4.1. *The non-homogeneous dydadic blocks Δ_j are defined by*

$$\Delta_j u = 0 \ \text{ if } \ j \leqslant -2, \quad \Delta_{-1} u = \widehat{\chi}(D) u = \int_{\mathbb{R}^n} \chi(x-y) u(y)\, dy,$$

and

$$\Delta_j u = \widehat{\varphi}(2^{-j} D) u = \int_{\mathbb{R}^n} 2^{jn} \varphi(2^j (x-y)) u(y)\, dy \ \text{ if } \ j \geqslant 0.$$

The non-homogeneous low frequency cut-off operators S_j are defined by

$$S_j u = \sum_{j' \leqslant j-1} \Delta_{j'} u.$$

Definition 4.2. *The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low frequency cut-off operators are define for all $j \in \mathbb{Z}$ by* [2]

$$\dot{\Delta}_j = \widehat{\varphi}(2^{-j} D) u = \int_{\mathbb{R}^n} 2^{jn} \varphi(2^j (x-y)) u(y)\, dy,$$

$$\dot{S}_j u = \widehat{\chi}(2^{-j} D) u = \int_{\mathbb{R}^n} 2^{jn} \chi(2^j (x-y)) u(y)\, dy.$$

Remark 4.1. *Observe that the dyadic blocks and low frequency cut-off operators are bounded maps from L^p into itself, and we will frequently make use of this property throughout this chapter.*

[2] Recall that $\sigma(D) u \doteq \mathcal{F}^{-1}(\sigma \widehat{u})$.

Obviously, we can write, at least formally, the Littlewood–Paley decompositions:

$$Id = \sum_j \Delta_j \text{ and } Id = \sum_j \dot{\Delta}_j. \tag{4.8}$$

In the non-homogeneous case, the above decomposition makes sense in $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 4.2. *Let u be in $\mathcal{S}'(\mathbb{R}^n)$. Then,*

$$u = \lim_{j\longrightarrow\infty} S_j u \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Proof. Note that $\langle u - S_j u, f\rangle = \langle u, f - S_j f\rangle$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Therefore, it suffices to prove that $f = \lim_{j\longrightarrow\infty} S_j f$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Because the Fourier transform is an automorphism of $\mathcal{S}(\mathbb{R}^n)$, we can alternatively prove that $\widehat{\chi}(2^{-j}\cdot)\widehat{f}$ converges to $\widehat{f}$ in $\mathcal{S}(\mathbb{R}^n)$, which follows easily. □

Another somewhat related result of convergence is the following.

Proposition 4.3. *Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of bounded functions such that the Fourier transform of u_j is supported in $2^j\tilde{\mathcal{C}}$, where $\tilde{\mathcal{C}}$ is a given annulus. Assume that, for some integer N, the sequence $(2^{-jN}\|u_j\|_{L^\infty})_{j\in\mathbb{N}}$ is bounded. The series $\sum_j u_j$ then converges in $\mathcal{S}'(\mathbb{R}^n)$.*

Proof. It turns out that for all integers j and k we may write

$$u_j = 2^{-jk} \sum_{|\alpha|=k} 2^{jn} g_\alpha(2^j\cdot) * \partial^\alpha u_j.$$

For any test function ϕ in $\mathcal{S}(\mathbb{R}^n)$, we then write

$$\langle u_j, \phi\rangle = 2^{-jk} \sum_{|\alpha|=k} \langle u_j, 2^{jn}\breve{g}_\alpha(2^j\cdot) * (-\partial)^\alpha \phi\rangle$$

with $\breve{g}_\alpha(x) = g_\alpha(-x)$. We then have

$$|\langle u_j, \phi\rangle| \leqslant C2^{-jk} \sum_{|\alpha|=k} 2^{jN} \|\partial^\alpha \phi\|_{L^1}.$$

Choose $k > N$, then $\sum_j \langle u_j, \phi\rangle$ is a convergent series, the sum of which is less than $C\|\phi\|_{M,\mathcal{S}}$ for some integer M. Thus, the forumla

$$\langle u, \phi\rangle \doteq \lim_{j\longrightarrow\infty} \sum_{j'\leqslant j} \langle u_{j'}, \phi\rangle$$

defines a tempered distribution. □

Proving the homogeneous Littlewood–Paley decomposition is more subtle. As discussed in the previous chapter, the decomposition does not hold for non-zero polynomials, however, it is true in $\mathcal{S}_0'(\mathbb{R}^n)$. Indeed, $\dot{S}_j u$ tends uniformly to 0 as $j \longrightarrow -\infty$.

Proposition 4.4. *Let $(u_j)_{j\in\mathbb{Z}}$ be a sequence of bounded functions such that the support of $\widehat{u}_j$ is included in $2^j\tilde{\mathcal{C}}$, where $\tilde{\mathcal{C}}$ is a given annulus. Assume that, for some integer N, the sequence $(2^{-jN}\|u_j\|_{L^\infty})_{j\in\mathbb{N}}$ is bounded and that the series $\sum_{j<0} u_j$ converges in L^∞. The series $\sum_{j\in\mathbb{Z}} u_j$ then converges to some u in $\mathcal{S}'(\mathbb{R}^n)$ and u belongs in $\mathcal{S}_0'(\mathbb{R}^n)$.*

Proof. By virtue of Proposition (4.3), the series $\sum_{j\in\mathbb{Z}} u_j$ converges to some u in $\mathcal{S}'(\mathbb{R}^n)$. We are therefore left with prove that u belongs to $\mathcal{S}_0'(\mathbb{R}^n)$. We have, for some integer N_0,

$$\|\dot{S}_j\|_{L^\infty} \leqslant \Big\|\dot{S}_j \sum_{j'\leqslant j+N_0} u_{j'}\Big\|_{L^\infty} \leqslant C\Big\|\sum_{j'\leqslant j+N_0} u_{j'}\Big\|_{L^\infty}.$$

As the series $\sum_{j<0} u_j$ converges in L^∞, the proposition is proved. □

4.2 Homogeneous Besov spaces

We start with a brief introduction to the homogeneous Besov spaces (including the related homogeneous Triebel–Lizorkin spaces) by defining such spaces and covering some of their most fundamental properties.

4.2.1 Introduction

Definition 4.3. *Let $s \in \mathbb{R}$ and $p, r \in [1,\infty]$. The homogeneous Besov space $\dot{B}^{s,r}_p(\mathbb{R}^n)$ consists of those distributions u in $\mathcal{S}_0'(\mathbb{R}^n)$ such that*

$$\|u\|_{\dot{B}^{s,r}_p} \doteq \left(\sum_{j\in\mathbb{Z}} (2^{sj}\|\dot{\Delta}_j u\|_{L^p})^r\right)^{\frac{1}{r}} < \infty.$$

Similarly, we can define the homogeneous Triebel–Lizorkin spaces.

Definition 4.4. *Let $s \in \mathbb{R}$ and $p, r \in [1,\infty]$. The homogeneous Triebel–Lizorkin space $\dot{F}^{s,r}_p(\mathbb{R}^n)$ consists of those distributions u in $\mathcal{S}_0'(\mathbb{R}^n)$ such that*

$$\|u\|_{\dot{F}^{s,r}_p} \doteq \left\|\left(\sum_{j\in\mathbb{Z}} (2^{sj}|\dot{\Delta}_j u|)^r\right)\right\|_{L^p} < \infty.$$

Proposition 4.5. *The spaces $\dot{B}^{s,r}_p(\mathbb{R}^n)$ and $\dot{F}^{s,r}_p(\mathbb{R}^n)$ endowed with $\|\cdot\|_{\dot{B}^{s,r}_p}$ and $\|\cdot\|_{\dot{F}^{s,r}_p}$, respectively, are normed spaces.*

Proof. We only prove this for the homogeneous Besov spaces since the same argument works for the homogeneous Triebel–Lizorkin spaces. Now, it is not too difficult to check that $\|\cdot\|_{\dot{B}^{s,r}_p}$ is a semi-norm. So, assume for some u in $\mathcal{S}'_0(\mathbb{R}^n)$, we have $\|u\|_{\dot{B}^{s,r}_p} = 0$. This implies that the support of $\widehat{u}$ is included in $\{0\}$. Thus, for any $j \in \mathbb{Z}$, we have $\dot{S}_j u = u$, but since u belongs in $\mathcal{S}'_0(\mathbb{R}^n)$, we have $u = 0$. This completes the proof. $\square$

Remark 4.2. *The definitions of the Besov space $\dot{B}^{s,r}_p(\mathbb{R}^n)$ and the Triebel–Lizorkin space $\dot{F}^{s,r}_p(\mathbb{R}^n)$ are independent of the test function φ used for defining the dyadic blocks $\dot{\Delta}_j$, and changing the function φ yields an equivalent norm. Namely, if $\tilde{\varphi}$ is another dyadic partition of unity, then an integer N_0 exists such that $|j - j'| \geqslant N_0$ implies that supp $\tilde{\varphi}(2^{-j}\cdot) \cap$ supp $\varphi(2^{-j}\cdot) = \varnothing$. Thus,*

$$\begin{aligned} 2^{sj}\|\tilde{\varphi}(2^{-j}D)u\|_{L^p} &= 2^{sj}\Big\| \sum_{|j-j'|\leqslant N_0} \tilde{\varphi}(2^{-j}D)\dot{\Delta}_{j'}u\Big\|_{L^p} \\ &\leqslant C2^{N_0|s|}\sum_{j'} \chi_{[-N_0,N_0]}(j-j')2^{sj'}\|\dot{\Delta}_{j'}u\|_{L^p}. \end{aligned}$$

Then, the result follows from Young's inequality.

As we have seen in our study of the Navier–Stokes equations in scale invariant function spaces, the homogeneous Besov and Triebel–Lizorkin spaces have nice scaling properties, which make them ideal spaces for examining global well-posedness for the Navier–Stokes equations. Indeed, if u is a tempered distribution and we consider the tempered distribution u_N where $u_N \doteq u(2^N\cdot)$, we get the following proposition.

Proposition 4.6. *Consider an integer N and a distribution $u \in \mathcal{S}'_0(\mathbb{R}^n)$ and set $u_N \doteq u(2^N\cdot)$. Then $\|u\|_{\dot{B}^{s,r}_p}$ is finite if and only if $\|u_N\|_{\dot{B}^{s,r}_p}$ is finite. Moreover, we have*

$$\|u_N\|_{\dot{B}^{s,r}_p} = 2^{N(s-n/p)}\|u\|_{\dot{B}^{s,r}_p}.$$

Remark 4.3. *As we have seen already, this proposition can be easily generalized to the following: there exists a $C > 0$, depending only on s, such that for all $\lambda > 0$, we have*

$$C^{-1}\lambda^{s-n/p}\|u\|_{\dot{B}^{s,r}_p} \leqslant \|u(\lambda\cdot)\|_{\dot{B}^{s,r}_p} \leqslant C\lambda^{s-n/p}\|u\|_{\dot{B}^{s,r}_p}.$$

Proof of Proposition (4.6)*.* By definition of $\dot{\Delta}_j$ and by the change of variable $z = 2^N y$, we get

$$\begin{aligned} \dot{\Delta}_j u_N(x) &= 2^{jn}\int_{\mathbb{R}^n} \varphi(2^j(x-y))u(2^N y)\,dy \\ &= 2^{(j-N)n}\int_{\mathbb{R}^n} \varphi(2^{j-N}(2^N x - z))u(z)\,dz \\ &= (\dot{\Delta}_{j-N}u)(2^n x). \end{aligned}$$

It turns out that $\|\dot{\Delta}_j u_N\|_{L^p} = 2^{-N\frac{n}{p}}\|\dot{\Delta}_{j-N}u\|_{L^p}$. We therefore conclude from this that

$$2^{sj}\|\dot{\Delta}_j u_N\|_{L^p} = 2^{N(s-\frac{n}{p})}2^{s(j-N)}\|\dot{\Delta}_{j-N}u\|_{L^p},$$

and the proposition follows immediately by summation. □

The following is an analogue to the Sobolev embedding theorem.

Proposition 4.7. *Let $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and $1 \leqslant r_1 \leqslant r_2 \leqslant \infty$. Then, for any real number s, the space $\dot{B}^{s,r_1}_{p_1}(\mathbb{R}^n)$ is continuously embedded in $\dot{B}^{s-n(\frac{1}{p_1}-\frac{1}{p_2}),r_2}_{p_2}(\mathbb{R}^n)$.*

Proof. From the Bernstein inequalities in Lemma 4.1,

$$\|\dot{\Delta}_j u\|_{L^{p_2}} \leqslant C2^{jn(\frac{1}{p_1}-\frac{1}{p_2})}\|\dot{\Delta}_j u\|_{L^{p_1}}.$$

The proposition follows from this and the fact that $l^{r_1}(\mathbb{Z})$ is continuously embedded in $l^{r_2}(\mathbb{Z})$. □

An interesting feature of homogeneous Besov spaces, in comparison with the Lebesgue and Sobolev spaces, is that they contain homogeneous functions of negative degree. For example, an earlier example showed the function $|x|^{-1}$, which is a homogeneous function of degree -1, belongs in $\dot{B}^{-1+n/p,\infty}_p(\mathbb{R}^n)$ but does not belong in $L^n(\mathbb{R}^n)$. In fact, we have the more general assertion.

Proposition 4.8. *Let $\sigma \in (0,n)$. For any $p \in [1,\infty]$, the function $|\cdot|^{-\sigma}$ belongs to $\dot{B}^{\frac{n}{p}-\sigma,\infty}_p(\mathbb{R}^n)$.*

Proof. Using Proposition 4.7, it enough to prove that $\rho_\sigma \doteq |\cdot|^{-\sigma}$ belongs to $\dot{B}^{n-\sigma,\infty}_1(\mathbb{R}^n)$. To do so, we introduce a smooth compactly supported function χ which identically equal to one near the unit ball and we write

$$\rho_\sigma = \rho_0 + \rho_1 \text{ with } \rho_0(x) \doteq \chi(x)|x|^{-\sigma} \text{ and } \rho_1(x) \doteq (1-\chi(x))|x|^{-\sigma}.$$

It is clear that $\rho_0 \in L^1(\mathbb{R}^n)$ and $\rho_1 \in L^q(\mathbb{R}^n)$ whenever $q > n/\sigma$. This implies that ρ_σ belongs to $\mathcal{S}'_0(\mathbb{R}^n)$. The homogeneity of the function ρ_σ then yields

$$\dot{\Delta}_j\rho_\sigma = 2^{jn}\rho_\sigma * \varphi(2^j\cdot) = 2^{j(n+\sigma)}\rho_\sigma(2^j\cdot) * \varphi(2^j\cdot) = 2^{j\sigma}(\dot{\Delta}_0\rho_\sigma)(2^j\cdot).$$

Therefore, $\|\dot{\Delta}_j\rho_\sigma\|_{L^1} = 2^{j(\sigma-n)}\|\dot{\Delta}_0\rho_\sigma\|_{L^1}$, which reduces the problem to proving that the function $\dot{\Delta}_0\rho_\sigma$ is in $L^1(\mathbb{R}^n)$. As ρ_0 is in $L^1(\mathbb{R}^n)$, $\dot{\Delta}_0\rho_0$ is also in $L^1(\mathbb{R}^n)$ due to the continuity of the operator $\dot{\Delta}_0$ on Lebesgue spaces. Using Lemma 4.1, we get

$$\|\dot{\Delta}_0\rho_1\|_{L^1} \leqslant C_k\|D^k\dot{\Delta}_0\rho_1\|_{L^1} \leqslant C_k\|D^k\rho_1\|_{L^1}.$$

By Leibniz's formula, $D^k\rho_1 - (1-\chi)D^k\rho_\sigma$ is a smooth compactly supported function. Then, choosing k so that $k > n-\sigma$ completes the proof of the proposition. □

Proposition 4.9. *A constant exists which satisfies the following properties. If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in (0,1)$, then we have, for any $p, r \in [1,\infty]$ and any $u \in \mathcal{S}_0'(\mathbb{R}^n)$,*

(i)
$$\|u\|_{\dot{B}_p^{\theta s_1+(1-\theta)s_2,r}} \leqslant \|u\|_{\dot{B}_p^{s_1,r}}^{\theta} \|u\|_{\dot{B}_p^{s_2,r}}^{1-\theta},$$

(ii)
$$\|u\|_{\dot{B}_p^{\theta s_1+(1-\theta)s_2,1}} \leqslant \frac{C}{s_2 - s_1}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right) \|u\|_{\dot{B}_p^{s_1,\infty}}^{\theta} \|u\|_{\dot{B}_p^{s_2,\infty}}^{1-\theta}.$$

Proof. To prove (i), we note that

$$2^{j(\theta s_1+(1-\theta)s_2)} \|\dot{\Delta}_j u\|_{L^p} = \left(2^{s_1 j}\|\dot{\Delta}_j u\|_{L^p}\right)^{\theta} \left(2^{s_2 j}\|\dot{\Delta}_j u\|_{L^p}\right)^{1-\theta}.$$

The result follows from Hölder's inequality.

To prove (ii), we estimate the low and high frequencies of u is a different way. Namely, we write

$$\|u\|_{\dot{B}_p^{\theta s_1+(1-\theta)\theta s_2,1}} = \sum_{j\leqslant N} 2^{j(\theta s_1+(1-\theta)s_2)}\|\dot{\Delta}_j u\|_{L^p} + \sum_{j>N} 2^{j(\theta s_1+(1-\theta)s_2)}\|\dot{\Delta}_j u\|_{L^p}.$$

By definition of the homogeneous Besov norms, we have

$$2^{j(\theta s_1+(1-\theta)s_2)}\|\dot{\Delta}_j u\|_{L^p} \leqslant 2^{j(1-\theta)(s_2-s_1)}\|u\|_{\dot{B}_p^{s_1,\infty}},$$

and

$$2^{j(\theta s_1+(1-\theta)s_2)}\|\dot{\Delta}_j u\|_{L^p} \leqslant 2^{-j(1-\theta)(s_2-s_1)}\|u\|_{\dot{B}_p^{s_2,\infty}},$$

Thus, we conclude that

$$\begin{aligned}\|u\|_{\dot{B}_p^{\theta s_1+(1-\theta)s_2,1}} &\leqslant \|u\|_{\dot{B}_p^{s_1,\infty}} \sum_{j\leqslant N} 2^{j(1-\theta)(s_2-s_1)} + \|u\|_{\dot{B}_p^{s_2,\infty}} \sum_{j>N} 2^{-j\theta(s_2-s_1)} \\ &\leqslant \|u\|_{\dot{B}_p^{s_1,\infty}} \frac{2^{N(1-\theta)(s_2-s_1)}}{2^{(1-\theta)(s_2-s_1)}-1} + \|u\|_{\dot{B}_p^{s_2,\infty}} \frac{2^{-N\theta(s_2-s_1)}}{1-2^{-\theta(s_2-s_1)}}.\end{aligned}$$

From this, if we choose N such that

$$\frac{\|u\|_{\dot{B}_p^{s_2,\infty}}}{\|u\|_{\dot{B}_p^{s_1,\infty}}} \leqslant 2^{N(s_2-s_1)} < 2^{s_2-s_1}\frac{\|u\|_{\dot{B}_p^{s_2,\infty}}}{\|u\|_{\dot{B}_p^{s_1,\infty}}},$$

we obtain inequality in (ii). □

The next lemma provides a useful criterion for determining whether the sum of a series belongs to a homogeneous Besov space.

Lemma 4.3. *Let $\mathcal{C}'$ be an annulus and $(u_j)_{j\in\mathbb{Z}}$ be a sequence of functions such that*

$$\text{supp } \widehat{u}_j \subset 2^j\mathcal{C}' \text{ and } \Big\|(2^{sj}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\Big\|_{l^r} < \infty.$$

If the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to some u in $\mathcal{S}'_0(\mathbb{R}^n)$, then u is in $\dot{B}^{s,r}_p(\mathbb{R}^n)$ and

$$\|u\|_{\dot{B}^{s,r}_p} \leqslant C(s)\Big\|(2^{sj}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\Big\|_{l^r}.$$

Remark 4.4. *The above convergence assumption concerns $(u_j)_{j<0}$. We note that if s, p and r satisfy the condition*

$$s < \frac{n}{p}, \text{ or } s = \frac{n}{p} \text{ and } r = 1, \tag{4.9}$$

then, by virtue of the Bernstein inequalities (cf. Lemma 4.1), we have

$$\lim_{j\longrightarrow-\infty} \sum_{j'<j} u_{j'} = 0 \text{ in } L^\infty(\mathbb{R}^n).$$

Hence, $\sum_{j\in\mathbb{Z}} u_j$ converges to some u in $\mathcal{S}'(\mathbb{R}^n)$, and $\dot{S}_j u \longrightarrow 0$ as $j \longrightarrow -\infty$. In particular, u belongs to $\mathcal{S}'(\mathbb{R}^n)$.

Proof of Lemma 4.3. It is clear that there exists some non-zero integer N_0 such that $\Delta_{j'}u_j = 0$ for $|j' - j| \geqslant N_0$. Thus,

$$\|\dot{\Delta}_j u\|_{L^p} = \Big\|\sum_{|j-j'|<N_0} \dot{\Delta}_{j'}u_j\Big\|_{L^p} \leqslant C \sum_{|j-j'|<N_0} \|u_j\|_{L^p}.$$

Therefore, we obtain that

$$2^{j's}\|\dot{\Delta}_{j'}u\|_{L^p} \leqslant C \sum_{|j-j'|\leqslant N_0} 2^{sj}\|u_j\|_{L^p}.$$

We deduce from this that

$$2^{sj}\|\dot{\Delta}_j u\|_{L^p} \leqslant \Big((c_k) * (d_l)\Big)_j \text{ with } c_k = C\chi_{[-N_0,N_0]}(k) \text{ and } d_l = 2^{sl}\|u_l\|_{L^p}.$$

Then, by Young's inequality (cf. Lemma 1.4 page 5 in [1]), we obtain

$$\|u\|_{\dot{B}^{s,r}_p} \leqslant C\Big\|(2^{sj}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\Big\|_{l^r}.$$

As u belongs to $\mathcal{S}'_0(\mathbb{R}^n)$, this proves the lemma. $\square$

The previous lemma allows us to establish the following important topological properties of homogeneous Besov spaces.

Theorem 4.1. *Let $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, r_1, r_2 \in [1, \infty]$. Assume that s_1, p_1 and r_1 satisfy (4.9). The space*

$$\dot{B}^{s_1,r_1}_{p_1}(\mathbb{R}^n) \cap \dot{B}^{s_2,r_2}_{p_2}(\mathbb{R}^n)$$

endowed with the norm $\|\cdot\|_{\dot{B}^{s_1,r_1}_{p_1}} + \|\cdot\|_{\dot{B}^{s_2,r_2}_{p_2}}$ is then complete and satisfies the Fatou property:

If $(u_i)_{i\in\mathbb{N}}$ is a bounded sequence in $\dot{B}^{s_1,r_1}_{p_1}(\mathbb{R}^n) \cap \dot{B}^{s_2,r_2}_{p_2}(\mathbb{R}^n)$, then there exist an element $u \in \dot{B}^{s_1,r_1}_{p_1}(\mathbb{R}^n) \cap \dot{B}^{s_2,r_2}_{p_2}(\mathbb{R}^n)$ and a subsequence $u_{\psi(i)}$ such that

(i) $\lim_{i\longrightarrow\infty} u_{\psi(i)} = u$ *in* $\mathcal{S}'(\mathbb{R}^n)$;

(ii) for $k = 1, 2$, $\|u\|_{\dot{B}^{s_k,r_k}_{p_k}} \leqslant C \liminf_{i\longrightarrow\infty} \|u_{\psi(i)}\|_{\dot{B}^{s_k,r_k}_{p_k}}$.

Proof. We first prove the Fatou property. By the Bernstein inequalities, for any $j \in \mathbb{Z}$, the sequence $(\dot{\Delta}_j u_n)_{n\in\mathbb{N}}$ is bounded in $L^{\min\{p_1,p_2\}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. By Cantor's diagonal argument, we can extract a subsequence $(u_{\psi(i)})_{n\in\mathbb{N}}$ and a sequence $(\tilde{u}_j)_{j\in\mathbb{Z}}$ of C^∞ functions with Fourier transform supported in $2^j\mathcal{C}$ such that, for any $j \in \mathbb{Z}$, $\phi \in \mathcal{S}$, and $k = 1, 2$,

$$\lim_{i\longrightarrow\infty} \langle \dot{\Delta}_j u_{\psi(i)}, \phi\rangle = \langle \tilde{u}_j, \phi\rangle \text{ and } \|\tilde{u}_j\|_{L^{p_k}} \leqslant \liminf_{i\longrightarrow\infty} \|\dot{\Delta}_j u_i\|_{L^{p_k}}.$$

The sequence,

$$\Big((2^{sj} \|\dot{\Delta}_j u_{\psi(i)}\|_{L^{p_k}})_j \Big)_{i\in\mathbb{N}},$$

is bounded in $l^{r_k}(\mathbb{Z})$. Hence, there exists an element $(\tilde{c}^k_j)_{j\in\mathbb{Z}}$ of $l^{r_k}(\mathbb{Z})$ for which (up to an omitted extraction) we have for any sequence $(d_j)_{j\in\mathbb{Z}}$ non-negative real numbers different from 0 for only a finite number of indices j,

$$\lim_{i\longrightarrow\infty} \sum_{j\in\mathbb{Z}} 2^{s_k j} \|\dot{\Delta}_j u_{\psi(i)}\|_{L^{p_k}} d_j = \sum_{j\in\mathbb{Z}} \tilde{c}^k_j d_j,$$

and

$$\|(\tilde{c}^k_j)_j\|_{l^{r_k}} \leqslant \liminf_{i\longrightarrow\infty} \|u_{\psi(i)}\|_{\dot{B}^{s_k,r_k}_{p_k}}.$$

Passing to the limit in the sum and using Lemma 1.2 page 2 of [1] with $X = \mathbb{Z}$ and μ the counting measure on $\mathbb{Z}$ gives that $(2^{s_k j}\|\tilde{u}_j\|_{L^{p_k}})_j$ belongs to $l^{r_k}(\mathbb{Z})$. From the definition of $\tilde{u}_j$, we conclude that $\mathcal{F}\tilde{u}_j$ is supported in the annulus $2^j\mathcal{C}$ where $\mathcal{C}$ is defined as in Proposition 4.1. As s_1, p_1 and r_1 satisfy (4.9), Lemma 4.3 guarantees that the series $\sum_{j\in\mathbb{Z}} \tilde{u}_j$ converges to some u in $\mathcal{S}'_0(\mathbb{R}^n)$. Given the property (4.3), we have, for all $M < N$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\langle \sum_{j=M}^{N} \dot{\Delta}_j u, \phi \right\rangle = \left\langle \sum_{j=M}^{N} \sum_{|j'-j|\leqslant 1} \dot{\Delta}_j \tilde{u}_{j'}, \phi \right\rangle.$$

Hence, by definition of $\tilde{u}_j$ and again by property (4.3), we have

$$\sum_{j=M}^{N} \dot{\Delta}_j u = \lim_{i\longrightarrow\infty} \sum_{j=M}^{N} \dot{\Delta}_j u_{\psi(i)} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Since the condition (4.9) is satisfies by s_1, p_1 and r_1 and $(u_{\psi(i)})_{n\in\mathbb{N}}$ is bounded in $\dot{B}^{s_1,r_1}_{p_1}(\mathbb{R}^n)$, Lemma 4.1 ensures that $\dot{S}_M u_{\psi(i)}$ tends uniformly to 0 when M goes to $-\infty$. Similarly, $(Id - \dot{S}_N)u_{\psi(i)}$ tends uniformly to 0 in, say, $\dot{B}^{s_2-1,r_2}_{p_2}(\mathbb{R}^n)$. Hence, u is indeed the limit of $(u_{\psi(i)})_{i\in\mathbb{N}}$ in $\mathcal{S}'(\mathbb{R}^n)$, which completes the proof of the Fatou property.

Now, we check $\dot{B}^{s_1,r_1}_{p_1}(\mathbb{R}^n) \cap \dot{B}^{s_2,r_2}_{p_2}(\mathbb{R}^n)$ is complete. Consider the Cauchy sequence $(u_i)_{i\in\mathbb{N}}$, which is of course bounded. Hence, there exists some u in $\dot{B}^{s_1,r_1}_{p_1}(\mathbb{R}^n) \cap \dot{B}^{s_2,r_2}_{p_2}(\mathbb{R}^n)$ and a subsequence $(u_{\psi(i)})_{i\in\mathbb{N}}$ which converges to u in $\mathcal{S}'(\mathbb{R}^n)$. Using the fact that for any $\epsilon > 0$, an integer $n(\epsilon)$ exists such that if $n \geqslant m \geqslant n(\epsilon)$, then

$$\|u_{\psi(m)} - u_{\psi(n)}\|_{\dot{B}^{s_1,r_1}_{p_1}} + \|u_{\psi(m)} - u_{\psi(n)}\|_{\dot{B}^{s_2,r_2}_{p_2}} < \epsilon,$$

and the Fatou property ensures that for all $m \geqslant n(\epsilon)$,

$$\|u_{\psi(m)} - u\|_{\dot{B}^{s_1,r_1}_{p_1}} + \|u_{\psi(m)} - u\|_{\dot{B}^{s_2,r_2}_{p_2}} < C\epsilon.$$

Hence, the subsequence $(u_{\psi(i)})_{i\in\mathbb{N}}$ tends to u in $\dot{B}^{s_1,r_1}_{p_1}(\mathbb{R}^n) \cap \dot{B}^{s_2,r_2}_{p_2}(\mathbb{R}^n)$ and this completes the proof of the theorem. □

Remark 4.5. *If $s > n/p$ or $s = n/p$ and $r > 1$, then $\dot{B}^{s,r}_p(\mathbb{R}^n)$ is no longer a Banach space. This is due to a breakdown of convergence for low frequencies, the so-called infrared divergence. There is a way to modify the definition of homogeneous Besov spaces so as to obtain a Banach space, regardless of the regularity index s. This is called realizing homogeneous Besov spaces. It turns out that realizations coincide with our definition when $s < n/p$ or $s = n/p$ and $r = 1$. In the other cases, however, realizations are defined up to a polynomial whose degree depends on $s - n/p$ and r. Unfortunately, dealing with partial differential equations in such spaces are quite difficult and tedious.*

For negative indices of regularity, i.e., $s < 0$, homogeneous Besov spaces may be characterized in terms of the low frequency cut-off operators $\dot{S}_j$.

Proposition 4.10. *Let $s < 0$, $p, r \in [1, \infty]$, and let u be a distribution in $\mathcal{S}'_0(\mathbb{R}^n)$. Then u belongs in $\dot{B}^{s,r}_p(\mathbb{R}^n)$ if and only if*

$$(2^{sj}\|\dot{S}_j u\|_{L^p})_{j\in\mathbb{Z}} \in l^r(\mathbb{Z}).$$

Moreover, for some constant C depending only on n, we have

$$C^{-|s|+1}\|u\|_{\dot{B}^{s,r}_p} \leqslant \left\|(2^{sj}\|\dot{S}_j u\|_{L^p})_j\right\|_{l^r} \leqslant C\left(1 + \frac{1}{|s|}\right)\|u\|_{\dot{B}^{s,r}_p}.$$

Proof. We write

$$\begin{aligned} 2^{sj}\|\dot{\Delta}_j u\|_{L^p} &\leqslant 2^{sj}(\|\dot{S}_{j+1}u\|_{L^p} + \|\dot{S}_j u\|_{L^p}) \\ &\leqslant 2^{-s}2^{s(j+1)}\|\dot{S}_{j+1}u\|_{L^p} + 2^{sj}\|\dot{S}_j u\|_{L^p}. \end{aligned}$$

This proves the left inequality. To prove the right inequality, we write

$$
\begin{aligned}
2^{sj}\|\dot{S}_j u\|_{L^p} &\leqslant 2^{sj}\sum_{j'\leqslant j-1}\|\dot{\Delta}_{j'}u\|_{L^p}\\
&\leqslant \sum_{j'\leqslant j-1} 2^{s(j-j')}2^{sj'}\|\dot{\Delta}_{j'}u\|_{L^p}.
\end{aligned}
$$

Since $s < 0$, the result follows by convolution. □

4.2.2 Comparison and inequalities between Besov and Lebesgue spaces

In what follows, we provide some useful embeddings of homogeneous Besov spaces into Lebesgue spaces, and vice versa. We omit proofs but refer readers to Chapter 2.5 in [1].

Proposition 4.11. *For any $p, q \in [1, \infty]$ such that $p \leqslant q$, the space $\dot{B}_p^{\frac{n}{p}-\frac{n}{q},1}(\mathbb{R}^n)$ is continuously embedded in the space $C_0(\mathbb{R}^n)$ of continuous functions vanishing at infinity. In addition, for all $q \in [1, \infty]$, the space $L^q(\mathbb{R}^n)$ is continuously embedded in the space $\dot{B}_q^{0,\infty}(\mathbb{R}^n)$, and the space $\mathcal{M}$ of bounded measures on $\mathbb{R}^n$ is continuously embedded in $\dot{B}_1^{0,\infty}(\mathbb{R}^n)$.*

The next theorem compares homogeneous Besov spaces with regularity index $s = 0$ and third index $r = 2$ to Lebesgue spaces.

Theorem 4.2. *For any $p \in [2, \infty)$, $\dot{B}_p^{0,2}(\mathbb{R}^n)$ is continuously included in $L^p(\mathbb{R}^n)$ and $L^{p'}(\mathbb{R}^n)$ where p' is the Hölder conjugate of p, is continuously included in $\dot{B}_{p'}^{0,2}(\mathbb{R}^n)$.*

Theorem 4.3. *For any $p \in [1, 2]$, the space $\dot{B}_p^{0,p}(\mathbb{R}^n)$ is continuously included in $L^p(\mathbb{R}^n)$, and $L^{p'}(\mathbb{R}^n)$ is continuously included in $\dot{B}_{p'}^{0,p'}(\mathbb{R}^n)$.*

The following theorem may be thought of as a refinement of the classical Sobolev embedding theorem.

Theorem 4.4. *Let $1 \leqslant p < q < \infty$ and let α be a positive real number. A constant C exists such that*

$$
\|f\|_{L^q} \leqslant C\|f\|_{\dot{B}_\infty^{-\alpha,\infty}}^{1-\theta}\|f\|_{\dot{B}_p^{s,p}}^{\theta},
$$

where $s = \alpha\left(\frac{q}{p} - 1\right)$ and $\theta = \frac{p}{q}$.

4.2.3 Heat flow characterization of homogeneous Besov and Triebel–Lizorkin spaces

For regularity index $s < 1$, there is a useful characterization of the Besov and Triebel–Lizorkin spaces using the heat semi-group operator $e^{t\Delta}$. In the following two propositions, the usual convention for when $p = \infty$ or $r = \infty$ in the norms should be understood.

Proposition 4.12. *Let $1 \leqslant p, r \leqslant \infty$ and $s < 1$, then the quantities*

$$\left(\sum_{j\in\mathbb{Z}} (2^{sj}\|\Delta_j u\|_p)^r\right)^{\frac{1}{r}}$$

and

$$\left(\int_0^\infty (t^{-s}\|e^{t\Delta}u\|_p)^r \, \frac{dt}{t}\right)^{\frac{1}{r}}$$

are equivalent and will be referred to by $\|u\|_{\dot{B}^{s,r}_p}$.

Proposition 4.13. *Let $1 \leqslant r \leqslant \infty$, $1 \leqslant p < \infty$ and $s < 1$, then the quantities*

$$\left\| \left(\sum_{j\in\mathbb{Z}} (2^{sj}|\Delta_j u|)^r\right)^{\frac{1}{r}} \right\|_p$$

and

$$\left\| \left(\int_0^\infty (t^{-s}|e^{t\Delta}u|)^r \, \frac{dt}{t}\right)^{\frac{1}{r}} \right\|_p$$

are equivalent and will be referred to by $\|u\|_{\dot{F}^{s,r}_p}$.

4.3 Homogeneous Paradifferential calculus

This section introduces the paradifferential calculus. Namely, we consider Bony's decomposition for the product of tempered distributions and how this product acts on homogeneous Besov spaces. Let u and v be tempered distributions in $\mathcal{S}'_0(\mathbb{R}^n)$. We have

$$u = \sum_{j'} \dot{\Delta}_{j'} u \;\text{ and }\; v = \sum_{j} \dot{\Delta}_j v,$$

then, at least formally,

$$uv = \sum_{j',j} \dot{\Delta}_{j'} u \dot{\Delta}_j v.$$

Paradifferential calculus is a mathematical tool for splitting the above sum into three parts.

- The first part concerns the indices (j', j) for which the size of supp $\mathcal{F}(\dot{\Delta}_{j'} u)$ is small compared to the size of supp $\mathcal{F}(\dot{\Delta}_j v)$, i.e., $j' \leqslant j - N_0$ for some suitable positive integer N_0.

- The second part contains indices corresponding to those frequencies of u which are large compared to the frequencies of v, i.e., $j' \geqslant j + N_0$.

- The last part we keep the indices (j, j') for which supp $\mathcal{F}(\dot{\Delta}_{j'}u)$ and supp $\mathcal{F}(\dot{\Delta}_j v)$ have comparable sizes, i.e., $|j - j'| \leqslant N_0$.

The suitable choice for N_0 depends on the assumptions made on the support of the function $\widehat{\varphi}$ used in the definition of the dyadic blocks. Hereafter, we shall always assume that φ is chosen according to Proposition 4.1 so that we have $N_0 = 1$. This leads to the following definition.

Definition 4.5. *The homogeneous paraproduct of v by u is defined as follows:*

$$\dot{T}_u v \doteq \sum_j \dot{S}_{j-1} u \dot{\Delta}_j v.$$

The homogeneous remainder of u and v is defined by

$$\dot{R}(u, v) \doteq \sum_{|k-j| \leqslant 1} \dot{\Delta}_k u \dot{\Delta}_j v.$$

Remark 4.6. *It can be checked that $\dot{T}_u v$ makes sense in $\mathcal{S}'(\mathbb{R}^n)$ whenever u and v are in $\mathcal{S}'_0(\mathbb{R}^n)$ and that $\dot{T} : (u, v) \longrightarrow \dot{T}_u v$ is a bilinear operator. Additionally, $\dot{R} : (u, v) \longrightarrow \dot{R}(u, v)$ is also a bilinear operator when restricted to sufficiently smooth distributions.*

The motivation for considering $\dot{T}$ and $\dot{R}$ is that, at least formally, the celebrated Bony decomposition holds, i.e.,

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v).$$

In order to understand how the product operates in Besov spaces, we need to study the continuity properties of the operators $\dot{T}$ and $\dot{R}$.

Remark 4.7. *To simplify the presentation, it should be understood hereafter that whenever expressions $\dot{T}_u v$ and $\dot{R}(u, v)$ appears, the series with general terms*

$$\dot{S}_{j-1} \dot{\Delta}_j v \quad \text{or} \quad \sum_{|\nu| \leqslant 1} \dot{\Delta}_j u \dot{\Delta}_{j-\nu} v$$

converges to some tempered distribution in $\mathcal{S}'_0(\mathbb{R}^n)$.

Theorem 4.5. *There exists a constant C such that for any $s \in \mathbb{R}$ and any $p, r \in [1, \infty]$, we have for any $(u, v) \in L^\infty(\mathbb{R}^n) \times \dot{B}^{s,r}_p(\mathbb{R}^n)$,*

$$\|\dot{T}_u v\|_{\dot{B}^{s,r}_p} \leqslant C^{1+|s|} \|u\|_{L^\infty} \|v\|_{\dot{B}^{s,r}_p}.$$

Moreover, for any $(s, t) \in \mathbb{R} \times (-\infty, 0)$ and any $p, r_1, r_2 \in [1, \infty]$,we have for any $(u, v) \in \dot{B}^{t,r_1}_\infty(\mathbb{R}^n) \times \dot{B}^{s,r_2}_p(\mathbb{R}^n)$,

$$\|\dot{T}_u v\|_{\dot{B}^{s+t,r}_p} \leqslant \frac{C^{1+|s+t|}}{-t} \|u\|_{\dot{B}^{t,r_1}_\infty} \|v\|_{\dot{B}^{s,r_2}_p}$$

with $\dfrac{1}{r} = \min\Big\{1, \dfrac{1}{r_1} + \dfrac{1}{r_2}\Big\}$.

Remark 4.8. *By virtue of Lemma 4.3 and the remark that follows it, the hypothesis of convergence is satisfied whenever* (s,p,r) *or* $(s+t,p,r)$ *satisfies* (4.9).

Proof of Theorem 4.5. According to (4.5), $\mathcal{F}(\dot{S}_{j-1}\dot{\Delta}_j v)$ is supported in $2^j\tilde{\mathcal{C}}$. Thus, we are left with proving the appropriate estimate for $\|\dot{S}_{j-1}\dot{\Delta}_j v\|_{L^p}$. Lemma 4.1 and Proposition 4.10 yield that for any $j \in \mathbb{Z}$ and $t < 0$,

$$\|\dot{S}_{j-1}u\|_{L^\infty} \leqslant C\|u\|_{L^\infty} \text{ and } \|\dot{S}_{j-1}u\|_{L^\infty} \leqslant \frac{C}{-t}c_{j,r_1}2^{-jt}\|u\|_{\dot{B}^{t,r_1}_{\infty}}, \tag{4.10}$$

where $(c_{j,r_1})_{j\in\mathbb{Z}}$ denotes an element of the unit sphere of $l^{r_1}(\mathbb{Z})$. Using Lemma 4.3, the estimates concerning the paraproducts are proved. □

Now we examine the behavior of the remainder operator $\dot{R}$; however, we have to consider terms of the type $\dot{\Delta}_j u\dot{\Delta}_j v$, the Fourier transforms of which are not supported in annuli, but rather in balls of the type $2^j B$. Thus, to prove that the remainder operator is bounded in Besov spaces, we need the following lemma.

Lemma 4.4. *Let* B *be a ball in* $\mathbb{R}^n$, s *is a positive real number, and* $p,r \in [1,\infty]$. *A constant* C *exists which satisfies the following. Let* $(u_j)_{j\in\mathbb{Z}}$ *be a sequence of smooth functions such that*

$$supp\ \hat{u}_j \subset 2^j B \ \ and \ \ \left\|(2^{sj}\|u_j\|_{L^p})\right\|_{l^r} < \infty.$$

We assume that the series $\sum_{j\in\mathbb{Z}} u_j$ *converges to* u *in* $\mathcal{S}'_0(\mathbb{R}^n)$. *We then have*

$$u \in \dot{B}^{s,r}_p(\mathbb{R}^n) \ \ and \ \ \|u\|_{\dot{B}^{s,r}_p} \leqslant \frac{C}{s}\left\|(2^{sj}\|u_j\|_{L^p})_j\right\|_{l^r(\mathbb{Z})}.$$

Remark 4.9. *Thanks to Lemma 4.4 and the remark that follows it, the hypothesis of convergence is satisfied whenever* s, p *and* r *satisfy condition* (4.9).

Proof of Lemma 4.4. As $\mathcal{C}$ is annulus and B is a ball, an integer N_1 exists such that if $j' \geqslant j + N_1$, then $2^{j'}\mathcal{C} \cap 2^j B = \varnothing$. So, if $j' \geqslant j + N_1$, then the Fourier transform of $\dot{\Delta}_{j'}u_j$, and thus $\Delta_{j'}u_j$, is equal to 0. Hence, we may write

$$\begin{aligned}\|\dot{\Delta}_{j'}u\|_{L^p} &\leqslant \sum_{j>j'-N_1}\|\dot{\Delta}_{j'}u_j\|_{L^p}\\ &\leqslant C\sum_{j>j'-N_1}\|u_j\|_{L^p}.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}2^{sj'}\|\dot{\Delta}_{j'}u\|_{L^p} &\leqslant \sum_{j>j'-N_1}2^{sj'}\|u_j\|_{L^p}\\ &\leqslant C\sum_{j>j'-N_1}2^{s(j'-j)}2^{sj}\|u_j\|_{L^p}.\end{aligned}$$

Since s is positive, applying Young's inequality for series completes the proof of the lemma. □

Remark 4.10. *Lemma 4.4 indeed fails if $s = 0$. To see this, fix a non-zero function $f \in L^p(\mathbb{R}^n)$ spectrally supported in some ball B, and a non-negative real α such that $\alpha r > 1$. Set $u_j = j^{-\alpha} f$ for $j \geqslant 1$, and $u_j = 0$ otherwise. It is clear that for all $j \in \mathbb{Z}$, supp $\widehat{u}_j \subset 2^j B$ and $\left\| (\|u_j\|_{L^p})_{j\in\mathbb{N}} \right\|_{l^r} < \infty$. If $r > 1$, then we can additionally set $\alpha < 1$ so that the series $\sum_j u_j$ diverges in $\mathcal{S}'(\mathbb{R}^n)$. If $r = 1$, then the series converges to a non-zero multiple of f. As $\dot{B}^{0,1}_p(\mathbb{R}^n)$ is a proper subspace of $L^p(\mathbb{R}^n)$, the function f need not be in $\dot{B}^{0,1}_p(\mathbb{R}^n)$, so the lemma also fails in this case.*

With the above lemma, we are ready to state and prove a resulting concerning the continuity of the remainder operator.

Theorem 4.6. *A constant C exists which satisfies the following inequalities. Let $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, r_1, r_2 \in [1, \infty]$. Assume that*

$$\frac{1}{p} \doteq \frac{1}{p_1} + \frac{1}{p_2} \leqslant 1 \ \text{ and } \ \frac{1}{r} \doteq \frac{1}{r_1} + \frac{1}{r_2} \leqslant 1.$$

If $s_1 + s_2$ is positive, then we have, for any $(u, v) \in \dot{B}^{s_1,r_1}_{p_1}(\mathbb{R}^n) \times \dot{B}^{s_2,r_2}_{p_2}(\mathbb{R}^n)$,

$$\|\dot{R}(u,v)\|_{\dot{B}^{s_1+s_2,r}_p} \leqslant \frac{C^{|s_1+s_2|+1}}{s_1+s_2} \|u\|_{\dot{B}^{s_1,r_1}_{p_1}} \|v\|_{\dot{B}^{s_2,r_2}_{p_2}}.$$

If $r = 1$ and $s_1 + s_2 \geqslant 0$, we have for any $(u, v) \in \dot{B}^{s_1,r_1}_{p_1}(\mathbb{R}^n) \times \dot{B}^{s_2,r_2}_{p_2}(\mathbb{R}^n)$,

$$\|\dot{R}(u,v)\|_{\dot{B}^{s_1+s_2,\infty}_p} \leqslant C^{|s_1+s_2|+1} \|u\|_{\dot{B}^{s_1,r_1}_{p_1}} \|v\|_{\dot{B}^{s_2,r_2}_{p_2}}.$$

Remark 4.11. *Thanks to Lemma 4.4 and the remark that follows it, the hypothesis of convergence is satisfied whenever $(s_1 + s_2, p, r)$ or $(s_1 + s_2, p, \infty)$ satisfies (4.9).*

Proof of Theorem 4.6. By definition of the homogeneous remainder operator,

$$\dot{R}(u,v) = \sum_j R_j \quad \text{where} \ \ R_j = \sum_{|\nu| \leqslant 1} \dot{\Delta}_{j-\nu} u \dot{\Delta}_j v.$$

Since $\widehat{\varphi}$ is supported in the annulus $\mathcal{C}$, the Fourier transform of R_j is supported in $2^j B_{24}(0)$. So, by construction of the dyadic partition of unity, there exists an integer N_0 such that

$$\text{if } j' > j + N_0, \ \text{ then } \ \dot{\Delta}_{j'} R_j = 0. \tag{4.11}$$

From this, we deduce that

$$\dot{\Delta}_{j'} \dot{R}(u,v) = \sum_{j \geqslant j' - N_0} \dot{\Delta}_{j'} R_j.$$

We deduce from Hölder's inequality that

$$\begin{aligned}
2^{(s_1+s_2)j'}\|\dot{\Delta}_{j'}\dot{R}(u,v)\|_{L^p} &\leqslant C2^{(s_1+s_2)j'}\sum_{|\nu|\leqslant 1, j\geqslant j'-N_0}\|\dot{\Delta}_{j-\nu}u\dot{\Delta}_j v\|_{L^p}\\
&\leqslant C2^{(s_1+s_2)j'}\sum_{|\nu|\leqslant 1, j\geqslant j'-N_0}\|\dot{\Delta}_{j-\nu}u\|_{L^{p_1}}\|\dot{\Delta}_j v\|_{L^{p_2}}\\
&\leqslant C2^{-(s_1+s_2)(j-j')}2^{s_1(j-\nu)}\sum_{|\nu|\leqslant 1, j\geqslant j'-N_0}\|\dot{\Delta}_{j-\nu}u\|_{L^{p_1}}2^{s_2 j}\|\dot{\Delta}_j v\|_{L^{p_2}}
\end{aligned}$$

In the case where $s_1+s_2 \geqslant 0$, we obtain the result by applying Hölder's inequality and Young's inequality for series to the above estimate. In the case where $r=1$ and s_1+s_2 is non-negative, we use the fact that

$$2^{(s_1+s_2)j'}\|\dot{\Delta}_{j'}\dot{R}(u,v)\|_{L^p} \leqslant C\sum_{|\nu|\leqslant 1, j\geqslant j'-N_0}2^{s_1(j-\nu)}\|\dot{\Delta}_{j-\nu}u\|_{L^{p_1}}2^{s_2 j}\|\dot{\Delta}_j v\|_{L^{p_2}},$$

then take the supremum over j' and use Hölder's inequality for series. □

With Bony's decomposition and the above basic results, we can derive a plethora of properties for Besov spaces. For instance, the space of bounded functions contained in homogeneous Besov spaces forms a ring structure.

Corollary 4.1. *If $s \in (0,\infty)$ and $p, r \in [1,\infty]$ satisfy (4.9), then $L^\infty(\mathbb{R}^n) \cap \dot{B}^{s,r}_p(\mathbb{R}^n)$ is an algebra. Moreover, there exists a constant C, depending only on the dimension n, such that*

Proof. Using Bony's decomposition, we have

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v).$$

According to Theorem 4.5, we have

$$\|\dot{T}_u v\|_{\dot{B}^{s,r}_p} \leqslant C^{s+1}\|u\|_{L^\infty}\|v\|_{\dot{B}^{s,r}_p} \text{ and } \|\dot{T}_v u\|_{\dot{B}^{s,r}_p} \leqslant C^{s+1}\|u\|_{\dot{B}^{s,r}_p}\|v\|_{L^\infty}.$$

Now, by applying Theorem 4.6 and using the fact that $L^\infty(\mathbb{R}^n) \hookrightarrow \dot{B}^{0,\infty}_\infty(\mathbb{R}^n)$, we get

$$\|\dot{R}(u,v)\|_{\dot{B}^{s,r}_p} \leqslant \frac{C^{s+1}}{s}\|u\|_{\dot{B}^{0,\infty}_\infty}\|v\|_{\dot{B}^{s,r}_p} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^{s,r}_p}.$$

This completes the proof of the corollary. □

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